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Citation: *AIP Conference Proceedings* **2041**, 050006 (2018); doi: 10.1063/1.5079375

View online: <https://doi.org/10.1063/1.5079375>

View Table of Contents: <http://aip.scitation.org/toc/apc/2041/1>

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Counter Concurrent Flows with General Conjugation Conditions

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Abstract. The article [1] is devoted to the conjugation problems for different physical processes such as the heat propagation in nonhomogeneous media, diffraction type problems, interaction of filtration and channel flows of a fluid, filtration in a borehole, reverse flows in a boundary layer after a separation point, etc. These conjugation problems are supplemented with examples of their realization including existence and uniqueness theorems of the corresponding boundary value problems. In the present article we consider counter concurrent flows with general conjugation conditions in the conjugation models and contact boundary value problems proposed.

INTRODUCTION

In the article we examine boundary value problems for forward-backward parabolic equations of the order $2n$ in the case of the full gluing matrix. As is known, in the case of boundary value problems for forward-backward parabolic equations, smoothness of initial and boundary data does not ensure the membership of a solution in Hölder spaces. Tersenov S.A. [2] in the simplest case exhibited necessary and sufficient conditions for solvability of these problems for parabolic second order equations in the spaces $H_{x,t}^{p,p/2}$ for $p > 2$. The solvability (orthogonality) conditions for the data of the problem in this case were written out explicitly. Note that the number of orthogonality conditions is finite in the one-dimensional case. At the same time the number of orthogonality conditions of an integral form is infinite in the multi-dimensional case [3].

Boundary value problems for counter concurrent flows in the linear case (mainly the model cases were considered) are examined in the articles by M.S. Bouendi, P. Grisvard, K.D. Pagani, G. Talenti, O. Arena, S.A. Tersenov, A.M. Nakhushiev, I.E. Egorov, A.A. Kerefov, N.V. Kislov, S.G. Pyatkov and other authors (see [2]–[5] and the bibliography therein). N.V. Kislov [6] studies similar problems with the use of the “projection theorem”, generalizing the corresponding result by M.S. Bouendi and P. Grisvard and S.G. Pyatkov relies on the properties of eigenfunctions of the corresponding eigenvalue problem. Nonlinear variable type equations are studied in [7], where the reader can find sufficiently complete bibliography as well.

CONJUGATION OF FLOWS

In a domain $(x, y) \in Q \subset \mathbb{R}^2$ we consider the equation

$$\mathbf{a}\nabla u = \operatorname{div}(\lambda\nabla u), \quad \mathbf{a} = \mathbf{a}(x, y, u), \quad \lambda = \lambda(x, y, u) \geq 0, \quad (1)$$

describing the propagation of the quantity u in a medium, more exactly transfer along the trajectories of the vector \mathbf{a} with the diffusion coefficient λ . For example, it is possible that u is the concentration of an admixture in a fluid flow, the temperature, or the density of a fluid.

The equation (1) also describes the flow of viscous incompressible fluid (the Stokes equations). In this case u is a component of the velocity vector $\mathbf{u} = (u, v)$ along the main direction Ox of a flow. The equation (1) in this case is

transformed to the form

$$\mathbf{a} \nabla u = \frac{\partial}{\partial y} (\lambda u_y), \quad \nabla u \equiv \nabla_{xy} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right), \quad u_y = \frac{\partial u}{\partial y}. \quad (2)$$

The formulas of the coordinate transformation from (x, y) to (ξ, η) under the rotation of the coordinate system around the origin by a positive angle φ are of the form $(\xi, \eta)^T = A(x, y)^T$, where $A = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}$.

Under this conditions, the equation (1) for $\lambda = 1$ is transformed to the form $\mathbf{Aa} \nabla_{\xi\eta} u = \nabla_{\xi\eta}^2 u$ and there by the equation (2) is written as

$$\mathbf{Aa} \nabla_{\xi\eta} u = u_{\eta\eta}, \quad u_{\eta\eta} = \frac{\partial^2 u}{\partial \eta^2}, \quad (3)$$

where

$$\nabla_{\xi\eta} = A \nabla_{xy}, \quad \mathbf{a} \nabla_{xy} = \mathbf{Aa} \nabla_{\xi\eta}, \quad \nabla_{\xi\eta}^2 = (A \nabla_{xy})^2 = \nabla_{xy}^2.$$

Let a curve Γ passing trough $(0, 0)$ divide the the domain Q into the domains Q_1 and Q_2 so that $Q = Q_1 \cup Q_2 \cup \Gamma$. The main directions of the propagation of the quantity $u(x, y)$ in Q_1 and Q_2 are the directions of the axes $0x$ and 0ξ , respectively. We assume that $u = u^1(x, y)$ and $u = u^2(\xi, \eta)$ in Q_1 and Q_2 satisfy the equations (2) and (3).

The continuity conditions of the functions u^k and their normal derivatives on Γ are of the form

$$u^1(\mathbf{x}) = u^2(A\mathbf{x}), \quad \nabla_{xy} u^1(\mathbf{x}) \cdot \mathbf{n} = A \nabla_{\xi\eta} u^2(\xi, \eta) \cdot \mathbf{n}|_{(\xi, \eta)^T = A(x, y)^T}, \quad (4)$$

where \mathbf{n} is the normal to Γ interior with respect to Q_1 .

COUNTER CONCURRENT FLOWS

In this case the conjugation conditions (4) are as follows:

$$u^1(\mathbf{x}) = u^2(A\mathbf{x}), \quad u_y^1(\mathbf{x}) = u_\eta^2(A\mathbf{x}), \quad \mathbf{x} \in \Gamma.$$

EQUATION OF THE SECOND ORDER

Let $Q = \Omega \times (0, T)$, where either Ω is a domain in \mathbb{R} or $\Omega \equiv \mathbb{R}$. We assume that $0 \in \Omega$ and the interval $\Gamma = (0, T)$ divides Q into two simply connected domains Q^+ and Q^- . In Q we consider the equation [2]

$$f(x)u_t = u_{xx}, \quad (5)$$

where

$$f(x) = \begin{cases} a, & x > 0 \quad (a = \text{const} > 0), \\ -b, & x < 0 \quad (b = \text{const} > 0). \end{cases}$$

A solution to the equation (5) satisfying the initial conditions

$$u(x, 0) = u_0(x) \quad (x \in \Omega^+), \quad u(x, T) = u_T(x) \quad (x \in \Omega^-) \quad (6)$$

and the following continuity conditions for a function and its first order derivatives:

$$u(-0, t) = u(+0, t), \quad u_x(-0, t) = u_x(+0, t) \quad (t \in (0, T)) \quad (7)$$

is unique in the class of bounded functions.

Theorem 1. Assume that $u_0, u_T \in H^p(\Omega^\pm)$, $p = 2l + \gamma$, $l \geq 1$ is an integer, $0 < \gamma < 2\theta \equiv \frac{2}{\pi} \arctan \sqrt{\frac{a}{b}}$ ($1 - 2\theta \leq \gamma < 1$) if $b \geq a$. Then under the $2l$ conditions

$$L_s(u_0, u_T) = 0, \quad s = 1, \dots, [p],$$

there exists a unique solution to the equation (5) in the Hölder space $H_x^{p, p/2}(Q^\pm)$ ($H_x^{2l+1-2\theta-2\varepsilon, l+\frac{1}{2}-\theta-\varepsilon}$), ε is an arbitrary small positive constant and $\varepsilon = 0$ for $\gamma > 1 - 2\theta$, which satisfies (6), (7).

EQUATION OF THE $2n$ ORDER

In the domain Q we consider the equation

$$f(x)u_t = Lu, \quad (8)$$

where L is a strictly elliptic operator of the order $2n$ in the variables x with Hölder coefficients in \bar{Q} .

Let $f(x) = \operatorname{sgn} x$ in (8). A solution to the equation (8) satisfying the initial conditions (6) and the continuity conditions for a solution and its derivatives up to and including the order $2n - 1$ of the form

$$\left. \frac{\partial^k u}{\partial x^k} \right|_{x=-0} = \left. \frac{\partial^k u}{\partial x^k} \right|_{x=+0}, \quad (k = 0, 1, \dots, 2n - 1) \quad (9)$$

is unique in the class of bounded functions.

By the method of simple layer parabolic potentials with unknown densities $\vec{\alpha}$ and $\vec{\beta}$ constructed with the help of a fundamental solution and the elementary Cattabriga solutions (see [8]), the boundary value problem (8), (6), (9) is reduced to solving the following systems of singular integral equations of the normal type:

$$K_1 \vec{\alpha} \equiv A \vec{\alpha}(t) + \frac{1}{\pi} \int_0^T \frac{B(T-t, T-\tau) \vec{\alpha}(\tau)}{\tau-t} d\tau = \vec{Q}_1(t), \quad (10)$$

$$K_2 \vec{\beta} \equiv A \vec{\beta}(t) - \frac{1}{\pi} \int_0^T \frac{B(t, \tau) \vec{\beta}(\tau)}{\tau-t} d\tau = \vec{Q}_2(t), \quad (11)$$

where A and B are matrices of the n -th order written out explicitly. Note that this representation is unique. The systems of singular integral equations

$$K_1 \alpha \equiv K_1^0 \alpha + k_1 \alpha = \vec{Q}_1, \quad K_2 \beta \equiv K_2^0 \beta + k_2 \beta = \vec{Q}_2$$

can be solved in the class of functions bounded at the ends of the interval $(0, T)$ (in the class $h(0, T)$ [9]) with the index $\alpha = -1$ [10, 11]. Regularizing the equations obtained we derive the systems of Fredholm equations

$$\vec{\alpha} + K_1^* k_1 \alpha = \vec{Q}_1^*, \quad \vec{\beta} + K_2^* k_2 \beta = \vec{Q}_2^*. \quad (12)$$

Solvability of the Fredholm equations (12) results from uniqueness of solutions to the basic problem (8), (6), (9) and their unique representations through the potentials.

If we look for a solution to (11) in the Hölder class $H^{p,p/2n}(Q^\pm)$ then for solvability of the problem (8), (6), (9) it is necessary that

$$L_s(u_0, u_T) = 0, \quad s = 1, \dots, 2 \left([p] - \frac{[p]}{2n} + 1 \right), \quad (13)$$

where L_s are integral operators containing the functions u_0, u_T .

Theorem 2. Let $u_0, u_T \in H^p(\Omega^\pm)$. Then under the conditions (13) there exists a unique solution to the equation (8) from the Hölder space $H^{p,p/2n}(Q^\pm)$ satisfying (6), (9).

EQUATION OF THE SIXTH ORDER. GENERAL CONJUGATION CONDITIONS

In the domain Q we consider the equation (8) for $n = 3$. We now describe general conjugation conditions and establish the dependence of exponents of the Hölder spaces on weighted conjugation functions [12, 13].

A solution to the system (8) is sought in the Hölder space $H_{x,t}^{p,p/6}(Q^+)$, $p = 6l + \gamma$, $0 < \gamma < 1$. It satisfies the initial conditions (6) and the conjugation conditions

$$\mathbf{u}(-0, t) = A \mathbf{u}(+0, t), \quad (14)$$

where $\mathbf{u}^k = (u^k, u_x^k, u_{xx}^k, u_{xxx}^k, u_{xxxx}^k, u_{xxxxx}^k)$, A is a nongenerate upper triangular matrix with constant real entries a_{ij} satisfying the condition

$$\begin{vmatrix} a_{45} & a_{46} \\ a_{55} & a_{56} \end{vmatrix} \neq 0, \quad \begin{vmatrix} a_{14} & a_{15} & a_{16} \\ a_{34} & a_{35} & a_{36} \\ a_{44} & a_{45} & a_{46} \end{vmatrix} \neq 0, \quad \begin{vmatrix} a_{24} & a_{25} & a_{26} \\ a_{34} & a_{35} & a_{36} \\ a_{44} & a_{45} & a_{46} \end{vmatrix} \neq 0. \quad (15)$$

We assume that the entries a_{ij} of A satisfy the uniqueness condition for solutions to the boundary value problem (8), (6), (14).

Theorem 3. Let the entries of the nondegenerate matrix A satisfy (15). Let also $\varphi_1(x), \varphi_2(x) \in H^p(\Omega^\pm)$ ($p = 6l + \gamma$). Then under the $12l - 4$ conditions (13), there exists a unique solution to (8) satisfying (6) and (9) from the space $H_x^{p,p/6}(Q^\pm)$.

CONCLUSION

The paper consider $2n$ -parabolic equations with changing time direction. For such problems smoothness of the initial and boundary data do not ensure the membership of a solution in some Hölder space. Application of the theory of singular equations along with the smoothness of the data of the problem makes it possible to find additional necessary and sufficient conditions ensuring that a solution belongs to the Hölder spaces $H_x^{p,p/2n}$ for $p \geq 2n$.

In the article we consider well-posedness questions of boundary value problems for parabolic equations of the sixth order with changing time direction with the full matrix of gluing conditions.

ACKNOWLEDGMENTS

The Work was Supported by the Ministry of Education and Science of Russia within the Framework of the State Task of Research for 2017-2019 (Project 1.6069.2017/8.9).

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