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SOLVABILITY OF NONLOCAL BOUNDARY VALUE PROBLEMS FOR PSEUDOPARABOLIC EQUATIONS

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We study nonlocal boundary value problems with the Samarkii and integral conditions for pseudoparabolic equations $u_t - a(x,t)u_{xx} + c(x,t)u - u_{xxt} = f(x,t)$. *The existence and uniqueness of a regular solution are established. Bibliography*: 9 *titles.*

Introduction

We study spatial-nonlocal boundary value problems for one-dimensional linear pseudoparabolic equations with the boundary condition combined from the nonlocal Samarskii boundary condition with variable coefficients and integral type boundary conditions. Such nonlocal problems for pseudoparabolic equations were earlier studied only in special cases (cf. [1, 2]). The method consists of 1) the passage from a problem for a "good" equation with "bad" boundary conditions to a problem with "good" boundary conditions, but for a "bad" equation," the so-called *loaded* equation [3, 4], 2) the proof of the solvability of the obtained problem by the parameter continuation method and a priori estimates, and 3) the construction of a solution to the original problem. Similar methods have been effectively used in close situations [5, 6].

1 Statement of the Problems

Suppose that Ω is the interval $(0,1)$ on the Ox-axis, Q is a rectangle $\Omega\times(0,T)$, $0<T<+\infty$, $a(x,t)$, $c(x,t)$, $K_1(x)$, $K_2(x)$, $f(x,t)$, $\alpha_1(t)$, $\alpha_2(t)$, $\beta_1(t)$, $\beta_2(t)$ are functions of $x\in\overline{\Omega}$, $t\in[0,T]$.

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Problem I. Find a solution $u(x,t)$ to the equation

$$Lu \equiv u_t - a(x,t)u_{xx} + c(x,t)u - u_{xxt} = f(x,t) \quad \text{in } Q \tag{1}$$

satisfying the conditions

$$u(x,0) = 0, \quad x \in \Omega, \tag{2}$$

$$u_x(0,t) = \alpha_1(t)u(0,t) + \alpha_2(t)u(1,t) + \int_0^1 K_1(x)u(x,t)\,dx, \quad 0 < t < T, \tag{3}$$

$$u_x(1,t) = \beta_1(t)u(0,t) + \beta_2(t)u(1,t) + \int_0^1 K_2(x)u(x,t)\,dx, \quad 0 < t < T. \tag{4}$$

Problem II. Find a solution $u(x,t)$ to Equation (1) in Q satisfying (2) and the conditions

$$u_x(0,t) = \alpha_1(t)u(0,t) + \alpha_2(t)u_x(1,t) + \int_0^1 K_1(x)u(x,t)\,dx, \quad 0 < t < T, \tag{5}$$

$$u(1,t) = \beta_1(t)u(0,t) + \beta_2(t)u_x(1,t) + \int_0^1 K_2(x)u(x,t)\,dx, \quad 0 < t < T. \tag{6}$$

Problem III. Find a solution $u(x,t)$ to Equation (1) in Q satisfying (2) and the conditions

$$u(0,t) = \alpha_1(t)u_x(0,t) + \alpha_2(t)u_x(1,t) + \int_0^1 K_1(x)u(x,t)\,dx, \quad 0 < t < T, \tag{7}$$

$$u(1,t) = \beta_1(t)u_x(0,t) + \beta_2(t)u_x(1,t) + \int_0^1 K_2(x)u(x,t)\,dx, \quad 0 < t < T. \tag{8}$$

Note that the function $\alpha_1(t)\beta_2(t) - \alpha_2(t)\beta_1(t)$ can vanish (in particular, identically) on $[0,T]$.

2 Solvability of Problem I

Introduce the space

$$V = \{v(x,t): \ v(x,t) \in L_\infty(0,T;W_2^2(\Omega)),\ v_t(x,t) \in L_2(0,T;W_2^1(\Omega)), v_{xxt}(x,t) \in L_2(Q)\}$$

equipped with the norm

$$\|v\|_V = \|v\|_{L_\infty(0,T;W_2^2(\Omega))} + \|v_t\|_{L_2(0,T;W_2^1(\Omega))} + \|v_{xxt}\|_{L_2(Q)}.$$

Suppose that necessary smoothness conditions (specified below) are satisfied. We set

$$\Delta_1(t) = \alpha_2(t) + \beta_2(t) - 2.$$

Assume that

$$\Delta_1(t) \neq 0, \quad t \in [0,T], \tag{9}$$

and set

$$\gamma_0(t) = \frac{1-\beta_2(t)}{\Delta_1(t)}, \quad \gamma_1(t) = \frac{\beta_2(t)-2}{\Delta_1(t)},$$

$$\delta_0(t) = \frac{\alpha_2(t)-1}{\Delta_1(t)}, \quad \delta_1(t) = -\frac{\alpha_2(t)}{\Delta_1(t)},$$

$$N_1(x,t) = \gamma_0(t)K_1(x) + \delta_0(t)K_2(x),$$

$$N_2(x,t) = \gamma_1(t)K_1(x) + \delta_1(t)K_2(x),$$

$$K(x,y,t) = x^2 N_1(y,t) + xN_2(y,t).$$

We define the operator B by the formula

$$(Bu)(x,t) = u(x,t) - \int\limits_{\Omega} K(x,y,t)u(y,t)\,dy.$$

For the sake of convenience, we preserve the notation $\overline{u}(x,t)$ for the action of the operator B on a function $u(x,t)$. Since the operator B is a Fredholm integral operator of the second kind with degenerate kernel, it is easy to write the invertibility conditions (the invertibility of B will be used below). We set

$$z_1(t) = 1 - \int\limits_{\Omega} x^2 N_1(x,t)\,dx, \quad z_2(t) = -\int\limits_{\Omega} xN_1(x,t)\,dx,$$

$$s_1(t) = -\int\limits_{\Omega} x^2 N_2(x,t)\,dx, \quad s_2(t) = 1 - \int\limits_{\Omega} xN_2(x,t)\,dx,$$

$$\Delta_{11}(t) = z_1(t)s_2(t) - z_2(t)s_1(t),$$

$$K_0(x,y,t) = \frac{1}{\Delta_{11}(t)}\{[x^2 s_2(t) - xs_1(t)]N_1(y,t) - [x^2 z_2(t) - xz_1(t)]N_2(y,t)\}.$$

In the case

$$\Delta_{11}(t) \neq 0, \quad t \in [0,T], \tag{10}$$

we have the equality

$$u(x,t) = \overline{u}(x,t) + \int\limits_{\Omega} K_0(x,y,t)\overline{u}(y,t)\,dy$$

which defines the inverse operator B^{-1}.

We introduce the function $\Phi(x,t,u)$ $(u = u(x,t))$ by the formula

$$\Phi(x,t,u) = \int\limits_{\Omega} K_{xx}(x,y,t)u_t(y,t)\,dy - \int\limits_{\Omega} K(x,y,t)u_{yyt}(y,t)\,dy - \int\limits_{\Omega} a(x,t)K(x,y,t)u_{yy}(y,t)\,dy$$

$$+ \int\limits_{\Omega} \{a(x,t)K_{xx}(x,y,t) - K_t(x,y,t) + K_{xxt}(x,y,t) - c(x,t)K(x,y,t) + c(y,t)K(x,y,t)\}u(y,t)\,dy.$$

Denote by $g(x,t)$ the function $\overline{f}(x,t)$. We consider the boundary value problem: find a solution $v(x,t)$ to the equation

$$Lv - \Phi(x,t,B^{-1}v) = g(x,t) \quad \text{in } Q \tag{11}$$

satisfying (2) and the conditions

$$v_x(0,t) = \alpha_1(t)v(0,t) + \alpha_2(t)v(1,t), \quad 0 < t < T, \tag{12}$$

$$v_x(1,t) = \beta_1(t)v(0,t) + \beta_2(t)v(1,t), \quad 0 < t < T. \tag{13}$$

Proposition 1. *Assume that the conditions* (9) *and* (10) *hold. Then if* $v(x,t)$ *is a solution in* V *to the boundary value problem* (11), (2), (12), (13), *then* $u = B^{-1}v$ *is a solution in* V *to Problem* I.

Proof. If $v(x,t)$ belongs to the space V then $u(x,t)$ belongs to the same space. The converse assertion is also valid. It is obvious that the conditions (2)–(4) are satisfied. We have the equality

$$B(Lu - f) = 0$$

from which, together with the condition (10) providing the one-to-one invertibility of the operator B, it follows that $u(x,t)$ is a solution to Equation (1). □

From Proposition 1 it is obvious that for proving the solvability of Problem I in V it suffices to establish the solvability of the boundary value problem (11), (2), (12), (13) in the same space.

Let λ be a number in $[0,1]$. We set

$$a_i(x,t,\lambda) = \frac{\lambda x^2}{2}[\beta_i(t) - \alpha_i(t)] + \lambda x \alpha_i(t), \quad i = 1,2,$$

$$b_1(x,t,\lambda) = a_1(x,t,\lambda) + \frac{a_2(x,t,\lambda)a_1(1,t,\lambda)}{1 - a_2(1,t,\lambda)},$$

$$b_2(x,t,\lambda) = \frac{a_2(x,t,\lambda)}{1 - a_2(1,t,\lambda)},$$

$$A_i(x,t,\lambda) = b_{ixx}(x,t,\lambda) - b_i(x,t,\lambda), \quad i = 1,2,$$

$$B_i(x,t,\lambda) = b_{ixxt}(x,t,\lambda) + a(x,t)b_{ixx}(x,t,\lambda) - c(x,t)b_i(x,t,\lambda) - b_{it}(x,t,\lambda), \quad i = 1,2,$$

$$F(x,t,\lambda,\xi) = A_1(x,t,\lambda)\xi_1 + A_2(x,t,\lambda)\xi_2 + B_1(x,t,\lambda)\xi_3 + B_2(x,t,\lambda)\xi_4,$$

$$\overline{w}(t) = (w_t(0,t), w_t(1,t), w(0,t), w(1,t)).$$

Theorem 1. *Assume that* (9) *and the following conditions hold*:

$$K_1(x) \equiv K_2(x) \equiv 0, \quad x \in \overline{\Omega}, \tag{14}$$

$$a(x,t) \in C^1(\overline{Q}), \quad c(x,t) \in C^1(\overline{Q}), \tag{15}$$

$$\alpha_i(t) \in C^1([0,T]), \quad \beta_i(t) \in C^1([0,T]), \quad i = 1,2; \tag{16}$$

$$c(x,t) \geqslant c_0 > 0, \quad (x,t) \in \overline{Q}, \tag{17}$$

$$[1 + 8\alpha_1(t)]\xi_1^2 + 8[\alpha_2(t) - \beta_1(t)]\xi_1\xi_2 + [1 - 8\beta_2(t)]\xi_2^2 \geqslant 0, t \in [0,T], \ (\xi_1,\xi_2) \in \mathbb{R}^2, \tag{18}$$

$$f(x,t) \in L_2(Q). \tag{19}$$

Then there exists a unique solution $u(x,t)$ *in* V *to Equation* (1) *in* Q *satisfying* (2), (12), (13).

Proof. We consider the auxiliary boundary value problem: find a solution $w(x,t)$ to the equation

$$L(\lambda)w \equiv w_t - aw_{xx} + cw - w_{xxt} - F(x,t,\lambda,\overline{w}) = f \tag{20}$$

in the rectangle Q that satisfies (2) and

$$w_x(0,t) = w_x(1,t) = 0, \quad 0 < t < T. \tag{21}$$

By the theorem about the parameter continuation method [7], for the solvability of the problem in the space V for all λ in $[0,1]$ and any function $f(x,t)$ in $L_2(Q)$, it suffices to prove

1) the continuity of the family of operators $\{L(\lambda)\}$ with respect to λ,

2) the solvability of the boundary value problem (20) (2), (21) in the space V for $\lambda = 0$,

3) an a priori estimate for all possible solutions $w(x,t)$ to the boundary value problem (20), (2), (21). in the space V, uniformly with respect to λ.

The continuity of $\{L(\lambda)\}$ with respect to λ is obvious. It is known [8, 9] that the problem (20), (2), (21) is solvable in the space V for $\lambda = 0$ under the conditions (15), (17), (19). Let us show that for all possible solutions $w(x,t)$ to the boundary value problem (20), (2), (21) an a priori estimate, uniform with respect to λ, holds in the space V.

Let $w(x,t)$ be a solution in V to the boundary value problem (20), (2), (21). We set

$$u(x,t) = w(x,t) + b_1(x,t,\lambda)w(0,t) + b_2(x,t,\lambda)w(1,t).$$

By the elementary inequalities

$$\begin{aligned} \int\limits_0^t v^2(0,\tau)\,d\tau &\leqslant \int\limits_0^t\int\limits_\Omega v_x^2(x,\tau)\,dx\,d\tau + 2\int\limits_0^t\int\limits_\Omega v^2(x,\tau)\,dx\,d\tau, \\ \int\limits_0^t v^2(1,\tau)\,d\tau &\leqslant \int\limits_0^t\int\limits_\Omega v_x^2(x,\tau)\,dx\,d\tau + 2\int\limits_0^t\int\limits_\Omega v^2(x,\tau)\,dx\,d\tau, \end{aligned} \tag{22}$$

it is easy to show that $u(x,t)$ belongs to the space V. A simple calculation shows that $u(x,t)$ is a solution to the equation

$$u_t - au_{xx} + cu - u_{xxt} = f \tag{23}$$

and satisfies (2) and the conditions

$$u_x(0,t) = \lambda[\alpha_1(t)u(0,t) + \alpha_2(t)u(1,t)], \quad 0 < t < T, \tag{24}$$

$$u_x(1,t) = \lambda[\beta_1(t)u(0,t) + \beta_2(t)u(1,t)], \quad 0 < t < T. \tag{25}$$

We consider the equality

$$\begin{aligned} &\int\limits_0^t\int\limits_\Omega (u_\tau - u_{xx\tau} + cu)\Big[u_\tau + \Big(x - \frac{1}{2}\Big)u_{x\tau} - u_{xx\tau}\Big]\,dx\,d\tau \\ &\qquad = \int\limits_0^t\int\limits_\Omega (f + au_{xx})\Big[u_\tau + \Big(x - \frac{1}{2}\Big)u_{x\tau} - u_{xx\tau}\Big]\,dx\,d\tau, \end{aligned} \tag{26}$$

which follows from Equation (23). Integrating by parts and using (2), (24), and (25), we obtain

$$\int_0^t\int_\Omega\left[\frac{1}{2}u_\tau^2+2u_{x\tau}^2+u_{xx\tau}^2\right]dx\,d\tau+\frac{1}{2}\int_\Omega c(x,t)[u^2(x,t)+u_x^2(x,t)]\,dx+\frac{1}{4}\int_0^t\{u_\tau^2(0,\tau)+u_\tau^2(1,\tau)$$
$$+8\lambda[\alpha_1(\tau)u_\tau^2(0,\tau)+[\alpha_2(\tau)-\beta_1(\tau)]u_\tau(0,\tau)u_\tau(1,\tau)-\beta_2(\tau)u_\tau^2(1,\tau)]\}\,d\tau$$
$$=\int_0^t\int_0^1\left(x-\frac{1}{2}\right)u_{x\tau}u_{xx\tau}dx\,d\tau+\int_0^t\int_\Omega(f+au_{xx})\left[u_\tau+\left(x-\frac{1}{2}\right)u_{x\tau}-u_{xx\tau}\right]dx\,d\tau$$
$$+\frac{1}{2}\int_0^t\int_\Omega c_\tau[u^2+u_x^2]\,dx\,d\tau-\int_0^t\int_\Omega\left(x-\frac{1}{2}\right)cuu_{x\tau}\,dx\,d\tau-\int_0^t\int_\Omega c_xuu_{x\tau}\,dx\,d\tau$$
$$-2\lambda\int_0^t\{u_\tau(0,\tau)[\alpha_1'(\tau)u(0,\tau)+\alpha_2'(\tau)u(1,\tau)]-u_\tau(1,\tau)[\beta_1'(\tau)u(0,\tau)+\beta_2'(\tau)u(1,\tau)]\}\,d\tau$$
$$+\lambda\int_0^t\{c(1,\tau)u(1,\tau)\big[\beta_1(\tau)u_\tau(0,\tau)+\beta_2(\tau)u_\tau(1,\tau)+\beta_1'(\tau)u(0,\tau)+\beta_2'(\tau)u(1,\tau)\big]$$
$$-c(0,\tau)u(0,\tau)\big[\alpha_1(\tau)u_\tau(0,\tau)+\alpha_2(\tau)u_\tau(1,\tau)+\alpha_1'(\tau)u(0,\tau)+\alpha_2'(\tau)u(1,\tau)\big]\}\,d\tau.$$

Using the conditions (15)–(18), the inequality (22), the Young inequality, and the obvious inequality

$$\int_0^1\int_\Omega u_{xx}^2\,dx\,dt\leqslant T\int_0^t\int_0^\tau\int_\Omega u_{xx\xi}^2\,dx\,d\xi\,d\tau, \tag{27}$$

we obtain the inequality

$$\int_0^t\int_\Omega(u_\tau^2+u_{x\tau}^2+u_{xx\tau}^2)\,dx\,d\tau+\int_\Omega[u^2(x,t)+u_x^2(x,t)]\,dx$$
$$\leqslant\delta\int_0^t\int_\Omega(u_\tau^2+u_{x\tau}^2+u_{xx\tau}^2)\,dx\,d\tau$$
$$+C_1\left[\int_0^t\int_\Omega(u^2+u_x^2)\,dx\,d\tau+\int_0^t\int_0^\tau\int_\Omega u_{xx\xi}^2\,dx\,d\xi\,d\tau+\int_0^t\int_\Omega f^2\,dx\,d\tau\right],$$

where δ is an arbitrary positive number, C_1 is determined by $a(x,t)$, $c(x,t)$, $\alpha_i(t)$, β_i, $i=1,2$, and the constants T and δ. We fix a small δ and, using the Gronwall lemma, find that the solution $u(x,t)$ to the problem (23), (2), (24), (25) satisfies the a priori estimate

$$\int_0^t\int_\Omega(u_\tau^2+u_{x\tau}^2+u_{xx\tau}^2)\,dx\,d\tau+\int_\Omega[u^2(x,t)+u_x^2(x,t)]\,dx\leqslant M_1,$$

where the constant M_1 depends only of the functions $a(x,t)$, $c(x,t)$, $\alpha_i(t)$, $\beta_i(t)$, $i=1,2$, and the number T. Combining this estimate with the inequality (26), we obtain the obvious estimate

$$\|u\|_V \leqslant M_2, \tag{28}$$

where the constant M_2 depends only on the functions $a(x,t)$, $c(x,t)$, $\alpha_i(t)$, $\beta_i(t)$, $i=1,2$, and the number T. Further, the representation

$$w(x,t) = u(x,t) - a_1(x,t,\lambda)u(0,t) - a_2(x,t,\lambda)u(1,t)$$

and the estimate (28) imply a similar estimate for the solution $w(x,t)$ to the problem (20), (2), (21). As was already mentioned, this estimate, the continuity of $\{L(\lambda)\}$ with respect to λ, and the solvability of the boundary value problem (20), (2), (21) in the space V for $\lambda=0$, imply the solvability of the boundary value problem (20), (2), (21) in the same space for $\lambda=1$. Denote by $w_1(x,t)$ the solution to the last problem. It is obvious that the function

$$u(x,t) = w_1(x,t) + b_1(x,t,1)w_1(0,t) + b_2(x,t,1)w_1(1,t),$$

is a required solution to the boundary value problem (1), (2), (12), (13).

The uniqueness of a solution is obvious. □

Consider the general case, i.e., the functions $K_1(x)$ and $K_2(x)$ do not vanish identically. Introduce the notation

$$k_1 = \max_{(x,t)\in\overline{Q}} \left[\int\limits_Q K_{xx}^2(x,y,t)\,dy \right], \quad k_2 = \max_{(x,t)\in\overline{Q}} \left[\int\limits_Q K^2(x,y,t)\,dy \right],$$

$$k_3 = \max_{(x,t)\in\overline{Q}} \left[\int\limits_Q K_0^2(x,y,t)\,dy \right], \quad k_4 = \max_{(x,t)\in\overline{Q}} \left[\int\limits_Q K_{0xx}^2(x,y,t)\,dy \right].$$

Theorem 2. *Assume that the conditions* (9), (15)–(19) *are satisfied. Let*

$$\exists\,\gamma_0 \in \left(0, \frac{\sqrt{7}}{2}\right) : \ \max\left\{ k_1(1+k_3)\left(6+\frac{1}{4\gamma_0^2}\right) + k_2k_4\left(6+\frac{1}{\gamma_0^2}\right),\ k_2\left(6+\frac{1}{\gamma_0^2}\right) \right\} < \frac{1}{4}. \tag{29}$$

Then there exists a unique solution $u(x,t)$ in V to Equation (1) *in Q satisfying* (2)–(4).

Proof. By Proposition 1, it suffices to establish the solvability of the problem (11), (2), (12), (13) in the space V. We again use the parameter continuation method: for $\lambda\in[0,1]$ and a given function $g(x,t)$ in the space $L_2(Q)$ we consider the following family of boundary value problems: find a solution $v(x,t)$ to the equation

$$Lv - \lambda\Phi(x,t,B^{-1}v) = g(x,t) \tag{30}$$

in the rectangle Q that satisfies the conditions (2), (12), (13). By Theorem 1, under the conditions (9), (15)–(19), the problem (30), (2), (12), (13) with $\lambda=0$ is solvable in V. Consequently, for the solvability of the problem (11), (2), (12), (13) in V it suffices to prove that all possible solutions to the problem (30), (2), (12), (13) satisfy an a priori estimate uniformly with respect to λ. We consider the equality

$$\int_0^t \int_\Omega (v_\tau - v_{xx\tau} + cv)\Big[v_\tau + \Big(x - \frac{1}{2}\Big)v_{x\tau} - v_{xx\tau}\Big]\,dx\,d\tau$$

$$= \int_0^t \int_\Omega [g + av_{xx} + \lambda\Phi(x,\tau,B^{-1}u)]\Big[v_\tau + \Big(x - \frac{1}{2}\Big)v_{x\tau} - v_{xx\tau}\Big]\,dx\,d\tau \tag{31}$$

which follows from Equation (30). In this equality, all the terms , except for the terms with $\Phi(x,\tau,B^{-1}u)$, are transformed in the same way as in the equality (26). Further,

$$\int_0^t \int_\Omega u_\tau^2\,dx\,d\tau \leqslant \frac{2(1+k_3)}{1-\delta_0^2}\int_0^t \int_\Omega v_\tau^2\,dx\,d\tau + m_1 \int_0^t \int_\Omega v^2\,dx\,d\tau, \tag{32}$$

$$\int_0^t \int_\Omega u_{xx\tau}^2\,dx\,d\tau \leqslant \frac{2}{1-\delta_0^2}\int_0^t \int_\Omega v_{xx\tau}^2\,dx\,d\tau + \frac{2k_4}{1-\delta_0^2}\int_0^t \int_\Omega v_\tau^2\,dx\,d\tau + m_2 \int_0^t \int_\Omega v^2\,dx\,d\tau, \tag{33}$$

$$\Big|\int_0^t \int_\Omega \Big(\int_\Omega K_{xx}(x,y,\tau)u_\tau(y,\tau)\,dy\Big)v_\tau(x,\tau)\,dx\,d\tau\Big|$$

$$\leqslant \Big[\frac{\delta_1^2}{2} + \frac{k_1(1+k_3)}{\delta_1^2(1-\delta_0^2)}\Big]\int_0^t \int_\Omega v_\tau^2\,dx\,d\tau + m_3 \int_0^t \int_\Omega v^2\,dx\,d\tau, \tag{34}$$

$$\Big|\int_0^t \int_\Omega \Big(\int_\Omega K_{xx}(x,y,\tau)u_\tau(y,\tau)\,dy\Big)\Big(x - \frac{1}{2}\Big)v_{x\tau}(x,\tau)\,dx\,d\tau\Big|$$

$$\leqslant \frac{\delta_2^2}{4}\int_0^t \int_\Omega v_{x\tau}^2\,dx\,d\tau + \frac{k_1(1+k_3)}{2\delta_2^2(1-\delta_0^2)}\int_0^t \int_\Omega v_\tau^2\,dx\,d\tau + m_4 \int_0^t \int_\Omega v^2\,dx\,d\tau, \tag{35}$$

$$\Big|\int_0^t \int_\Omega \Big(\int_\Omega K_{xx}(x,y,\tau)u_\tau(y,\tau)\,dy\Big)v_{xx\tau}(x,\tau)\,dx\,d\tau\Big|$$

$$\leqslant \frac{\delta_3^2}{2}\int_0^t \int_\Omega v_{xx\tau}^2\,dx\,d\tau + \frac{k_1(1+k_3)}{\delta_3^2(1-\delta_0^2)}\int_0^t \int_\Omega v_\tau^2\,dx\,d\tau + m_5 \int_0^t \int_\Omega v^2\,dx\,d\tau, \tag{36}$$

$$\Big|\int_0^t \int_\Omega \Big(\int_\Omega K(x,y,\tau)u_{yy\tau}(y,\tau)\,dy\Big)v_\tau(x,\tau)\,dx\,d\tau\Big|$$

$$\leqslant \Big[\frac{\delta_4^2}{2} + \frac{k_2k_4)}{\delta_4^2(1-\delta_0^2)}\Big]\int_0^t \int_\Omega v_\tau^2\,dx\,d\tau + \frac{k_2}{\delta_4^2(1-\delta_0^2)}\int_0^t \int_\Omega v_{xx\tau}^2\,dx\,d\tau + m_6 \int_0^t \int_\Omega v^2\,dx\,d\tau, \tag{37}$$

$$\left|\int_0^t\int_\Omega\left(\int_\Omega K(x,y,\tau)u_{yy\tau}(y,\tau)\,dy\right)\left(x-\frac{1}{2}\right)v_{x\tau}(x,\tau)\,dx\,d\tau\right| \leqslant \frac{\delta_5^2}{2}\int_0^t\int_\Omega v_{x\tau}^2\,dx\,d\tau$$

$$+\frac{k_2k_4}{\delta_5^2(1-\delta_0^2)}\int_0^t\int_\Omega v_\tau^2\,dx\,d\tau+\frac{k_2}{\delta_5^2(1-\delta_0^2)}\int_0^t\int_\omega v_{xx\tau}^2\,dx\,d\tau+m_7\int_0^t\int_\Omega v^2\,dx\,d\tau, \tag{38}$$

$$\left|\int_0^t\int_\Omega\left(\int_\Omega K(x,y,\tau)u_{yy\tau}(y,\tau)\,dy\right)v_{xx\tau}(x,\tau)\,dx\,d\tau\right|$$

$$\leqslant\left[\frac{\delta_6^2}{2}+\frac{k_2}{\delta_6^2(1-\delta_0^2)}\right]\int_0^t\int_\Omega v_{xx\tau}^2\,dx\,d\tau+\frac{k_2k_4}{\delta_6^2(1-\delta_0^2)}\int_0^t\int_\Omega v_\tau^2\,dx\,d\tau+m_8\int_0^t\int_\Omega v^2\,dx\,d\tau, \tag{39}$$

where δ_0–δ_6 are arbitrary positive numbers, m_1–m_8 are determined by the functions $\alpha_i(t)$, $\beta_i(t)$, $K_i(x)$, $i=1,2$, and the number δ_0–δ_6 (the inequalities (32) and (33) are proved with the help of the representation of $u(x,t)$ via $v(x,t)$ and the Hölder and Young inequalities. The inequalities (34)–(39) are obtained by using the Hölder and Young inequalities and the inequalities (32) and (33). By the obtained inequality, the Hölder and Young inequalities, the inequalities (22), (27), and the conditions (2), (12), (13), (15)–(19), it is easy to obtain from (31) to the inequality

$$\int_0^t\int_\Omega\left[\frac{1}{2}v_\tau^2+2v_{x\tau}^2+v_{xx\tau}^2\right]dx\,d\tau+\frac{c_0}{2}\int_\Omega[v^2(x,t)+v_x^2(x,t)]\,dx$$

$$\leqslant\left[\frac{\delta_1^2}{2}+\frac{k_1(1+k_3)}{\delta_1^2(1-\delta_0^2)}+\frac{k_1(1+k_3)}{2\delta_2^2(1-\delta_0^2)}+\frac{k_1(1+k_3)}{\delta_3^2(1-\delta_0^2)}+\frac{\delta_4^2}{2}+\frac{k_2k_4}{\delta_4^2(1-\delta_0^2)}\right.$$

$$\left.+\frac{k_2k_4}{\delta_5^2(1-\delta_0^2)}+\frac{k_2k_4}{\delta_6^2(1-\delta_0^2)}+\delta\right]\int_0^t\int_\Omega v_\tau^2\,dx\,d\tau+\left[\frac{\delta_2^2}{4}+\frac{\delta_5^2}{2}+\delta\right]\int_0^t\int_\Omega v_{x\tau}^2\,dx\,d\tau$$

$$+\left[\frac{\delta_3^2}{2}+\frac{k_2}{\delta_4^2(1-\delta_0^2)}+\frac{k_2}{\delta_5^2(1-\delta_0^2)}+\frac{\delta_6^2}{2}+\frac{k_2}{\delta_6^2(1-\delta_0^2)}+\delta\right]\int_0^t\int_\Omega v_{xx\tau}^2\,dx\,d\tau$$

$$+m_9\left[\int_0^t\int_\Omega(v^2+v_x^2)\,dx\,d\tau+\int_0^t\int_0^\tau\int_\Omega v_{xx\xi}^2\,dx\,d\xi\,d\tau+\int_0^t\int_\Omega g^2\,dx\,d\tau\right], \tag{40}$$

where δ is an arbitrary positive number and m_9 is determined by the functions $a(x,t)$, $c(x,t)$, $\alpha_i(t)$, $\beta_i(t)$, $K_i(x)$, $i=1,2$, and the numbers T, δ, δ_0–δ_6. We fix δ_1–δ_6 by setting $\delta_1=\delta_4=1/2$, $\delta_2=\sqrt{2}\gamma_0$, $\delta_5=\gamma_0$, $\delta_3=\delta_6=1/\sqrt{2}$. Now, for sufficiently small fixed δ and δ_0, using the condition (29), we get

$$\int_0^t\int_\Omega(v_\tau^2+v_{x\tau}^2+v_{xx\tau}^2)\,dx\,d\tau+\int_\Omega[v^2(x,t)+v_x^2(x,t)]\,dx$$

$$\leqslant m_0\left[\int_0^t\int_\Omega(v^2+v_x^2)\,dx\,d\tau+\int_0^t\int_0^\tau\int_\Omega v_{xx\xi}^2\,dx\,d\xi\,d\tau+\int_0^t\int_\Omega g^2\,dx\,d\tau\right].$$

This inequality and the Gronwall lemma imply that the solution to the boundary value problem (30), (2), (12), (13) satisfies the a priori estimate

$$\|v\|_V \leqslant M,$$

where the constant M is defined by the functions $a(x,t)$, $c(x,t)$, $\alpha_i(t)$, $\beta_i(t)$, $K_i(x)$, $i=1,2$, and the number T. As was already mentioned, this estimate implies the solvability of the problem (11), (2), (12), (13) in V. As above, taking into account Proposition 1, we obtain the solvability of Problem I in the space V. The uniqueness of a solution is obvious. □

3 Solvability of Problem II

As above, we assume that the required smoothness conditions are satisfied. We set

$$\Delta_2(t) = [1-\alpha_2(t)][1-\beta_1(t)] + \alpha_1(t)[1-\beta_2(t)].$$

Assume that

$$\Delta_2(t) \neq 0, \quad t \in [0,T]. \tag{41}$$

We set

$$\mu_0(t) = \frac{1-\beta_1(t)}{\Delta_2(t)}, \quad \nu_0(t) = \frac{\alpha_1(t)}{\Delta_2(t)}, \quad \mu_1(t) = \frac{\beta_2(t)-1}{\Delta_2(t)}, \quad \nu_1(t) = \frac{1-\alpha_2(t)}{\Delta_2(t)},$$
$$R_1(x,t) = \mu_0(t)K_1(x) + \nu_0(x)K_2(x), \quad R_2(x,t) = \mu_1(t)K_1(x) + \nu_1(x)K_2(x),$$
$$R(x,y,t) = xR_1(y,t) + R_2(y,t),$$
$$\overline{z}_1(t) = 1 - \int\limits_\Omega xR_1(x,t)\,dx, \quad \overline{z}_2(t) = -\int\limits_\Omega R_1(x,t)\,dx,$$
$$\overline{s}_1(t) = -\int\limits_\Omega xR_2(x,t)\,dx, \quad \overline{s}_2(t) = 1 - \int\limits_\Omega R_2(x,t)\,dx,$$
$$\Delta_{21}(t) = \overline{z}_1(t)\overline{s}_2(t) - \overline{z}_2(t)\overline{s}_1(t),$$
$$R_0(x,y,t) = \frac{1}{\Delta_{21}(t)}\{[x\overline{s}_2(t) - \overline{s}_1(t)]R_1(y,t) - [x\overline{z}_2(t) - \overline{z}_1(t)]R_1(y,t)\}.$$

We again introduce the integral operator B by the formula

$$(Bu)(x,t) = u(x,t) - \int\limits_\Omega R(x,y,t)u(y,t)\,dy$$

under the condition

$$\Delta_{21}(t) \neq 0, \quad t \in [0,T]. \tag{42}$$

This operator is invertible, and the operator B^{-1} is defined by the formula

$$(B^{-1}v)(x,t) = v(x,t) + \int\limits_\Omega R_0(x,y,t)v(y,t)\,dy.$$

We introduce the function $\Psi(x,t,u)$ by the formula

$$\Psi(x,t,u) = \int\limits_\Omega R_{xx}(x,y,t)u_t(y,t)\,dy - \int\limits_\Omega R(x,y,t)u_{yyt}(y,t)\,dy$$
$$- \int\limits_\Omega a(x,t)R(x,y,t)u_{yy}(y,t)\,dy + \int\limits_\Omega \{a(x,y)R_{xx}(x,y,t) - R_t(x,y,t)$$
$$+ R_{xxt}(x,y,t) - c(x,t)R(x,y,t) + c(y,t)R(x,y,t)\}u(y,t)\,dy.$$

We consider the boundary value problem: find a solution $v(x,t)$ to the equation

$$Lv - \Psi(x,t,B^{-1}v) = \overline{f}(x,t) \quad \text{in } Q \tag{43}$$

that satisfies (2) and the conditions

$$v_x(0,t) = \alpha_1(t)v(0,t) + \alpha_2(t)v_x(1,t), \quad 0 < t < T, \tag{44}$$

$$v(1,t) = \beta_1(t)v(0,t) + \beta_2(t)v_x(1,t), \quad 0 < t < T. \tag{45}$$

Proposition 2. *Suppose that (41) and (42) hold. Then if function $v(x,t)$ is a solution in V to the boundary value problem* (43), (2), (44), (45), *then the function $u = B^{-1}v$ is a solution to Problem* II *in V.*

The proof is similar to that of Proposition 1.

Theorem 3. *Suppose that* (14)–(17), (19), *(41), and the following conditions hold*:

$$8\alpha_1(t)\xi_1^2 + 8[\alpha_2(t) - \beta_1(t)]\xi_1\xi_2 + [1 - 8\beta_2(t)]\xi_2^2 \geqslant 0$$

for

$$t \in [0,T], \quad (\xi_1,\xi_2) \in \mathbb{R}^2; \tag{46}$$

$$f(x,t) \in L_2(Q). \tag{47}$$

Then there exists a unique function $u(x,t)$ in V that is a solution to Equation (1) *in Q satisfying the conditions* (2), (44), *and* (45).

Proof. We again use the parameter continuation method. Let $\lambda \in [0,1]$. We set

$$\overline{a}_i(x,t,\lambda) = \lambda[x\alpha_i(t) + \beta_i(t) - \alpha_i(t)], \quad i = 1,2,$$
$$\Delta(t,\lambda) = [1 - a_1(0,t,\lambda)][1 - a_{2x}(1,t,\lambda)] - a_{1x}(1,t,\lambda)a_2(0,t,\lambda),$$
$$\overline{b}_1(x,t,\lambda) = \frac{1}{\Delta(t,\lambda)}\{a_1(x,t,\lambda)[1 - a_{2x}(1,t,\lambda)] + a_2(x,t,\lambda)a_1(1,t,\lambda)\},$$
$$\overline{b}_2(x,t,\lambda) = \frac{1}{\Delta(t,\lambda)}\{a_1(x,t,\lambda)a_2(0,t,\lambda) + a_2(x,t,\lambda)[1 - a_1(0,t,\lambda)]\},$$
$$\overline{A}_i(x,t,\lambda) = \overline{b}_{ixx}(x,t,\lambda) - \overline{b}_i(x,t,\lambda), \quad i = 1,2,$$
$$\overline{B}_i(x,t,\lambda) = \overline{b}_{ixxt}(x,t,\lambda) + a(x,t)\overline{b}_{ixx}(x,t,\lambda) - c(x,t)\overline{b}_i(x,t,\lambda) - \overline{b}_{it}(x,t,\lambda), \quad i = 1,2,$$
$$\overline{F}(x,t,\lambda,\xi) = \overline{A}_1(x,t,\lambda)\xi_1 + \overline{A}_2(x,t,\lambda)\xi_2 + \overline{B}_1(x,t,\lambda)\xi_3 + \overline{B}_2(x,t,\lambda)\xi_4,$$
$$\overline{w}(t) = (w_t(0,t), w_{xt}(1,t), w(0,t), w_x(1,t)).$$

We consider the following family of auxiliary boundary value problems: find a function $w(x,t)$ that is a solution to the equation

$$L(\lambda)w \equiv w_t - aw_{xx} + cw - w_{xxt} - \overline{F}(x,t,\lambda,\overline{w}) = f \tag{48}$$

in the rectangle Q and satisfies (2) and the conditions

$$w_x(0,t) = w(1,t) = 0, \quad 0 < t < T. \tag{49}$$

For this family of problems, the continuity of the operator $L(\lambda)$ with respect to λ is obvious. The solvability of the problem (48), (2), (49) in the space V for $\lambda = 0$ is known [8, 9]. It remains to prove an a priori uniform estimate λ for the solution in the space V.

We introduce the function $u(x,t) = w(x,t) + \overline{b}_1(x,t,\lambda)w(0,t) + \overline{b}_2(x,t,\lambda)w_x(1,t)$. It is obvious that if $w(x,t)$ belongs to V, then $u(x,t)$ belongs to the same space and the function $u(x,t)$ is a solution to the equation

$$u_t - au_{xx} + cu - u_{xxt} = f \tag{50}$$

satisfying (2) and the condition

$$u_x(0,t) = \lambda[\alpha_1(t)u(0,t) + \alpha_2(t)u_x(1,t)], \quad 0 < t < T, \tag{51}$$

$$u(1,t) = \lambda[\beta_1(t)u(0,t) + \beta_2(t)u_x(1,t)], \quad 0 < t < T. \tag{52}$$

We consider the equality

$$\begin{aligned}
&\int_0^t \int_\Omega (u_\tau - u_{xx\tau} + cu)[u_\tau - \Big(x - \frac{1}{2}\Big)u_{x\tau} - u_{xx\tau}]\,dx\,d\tau \\
&= \int_0^t \int_\Omega (f + au_{xx})[u_\tau - \Big(x - \frac{1}{2}\Big)u_{x\tau} - u_{xx\tau}]\,dx\,d\tau
\end{aligned} \tag{53}$$

following from (50). Integrating by parts and using (2), (51), (52), it is easy to transform this equality to the form

$$\begin{aligned}
&\int_0^t \int_\Omega [u_\tau^2 + \frac{3}{2}u_{x\tau}^2 + u_{xx\tau}^2]\,dx\,d\tau + \frac{1}{2}\int_\Omega c(x,t)[u^2(x,t) + u_x^2(x,t)]\,dx + \frac{1}{4}\int_0^t u_{x\tau}^2(0,\tau)\,d\tau \\
&+ \frac{1}{4}\int_0^t \{8\lambda\alpha_1(\tau)u_\tau^2(0,\tau) + 8\lambda[\alpha_2(\tau) - \beta_1(\tau)]u_\tau(0,\tau)u_{x\tau}(1,\tau) + [1 - 8\lambda\beta_2(\tau)]u_{x\tau}^2(1,\tau)\}\,d\tau \\
&= \int_0^t \int_\Omega (f + au_{xx})[u_\tau - \Big(x - \frac{1}{2}\Big)u_{x\tau} - u_{xx\tau}]\,dx\,d\tau - \int_0^t \int_\Omega c_x u u_{x\tau}\,dx\,d\tau \\
&+ \frac{1}{2}\int_0^t \int_\Omega c_\tau(u^2 + u_x^2)\,dx\,d\tau + \int_0^t \int_\Omega \Big(x - \frac{1}{2}\Big)u_\tau u_{x\tau}\,dx\,d\tau \\
&+ \int_0^t \int_\Omega \Big(x - \frac{1}{2}\Big)cuu_{x\tau}\,dx\,d\tau - 2\lambda\int_0^t u_\tau(0,\tau)[\alpha_1'(\tau)u(0,\tau) + \alpha_2'(\tau)u_x(1,\tau)]
\end{aligned}$$

$$- u_{x\tau}(1,\tau)[\beta_1'(\tau)u(0,\tau) + \beta_2'(\tau)u_x(1,\tau)]\,d\tau + \lambda \int\limits_0^t c(1,\tau)u(1,\tau)u_{x\tau}(1,\tau)\,d\tau$$

$$- \lambda \int\limits_0^t c(0,\tau)u(0,\tau)[\alpha_1(\tau)u_\tau(0,\tau) + \alpha_2(\tau)u_{x\tau}(1,\tau) + \alpha_1'(\tau)u(0,\tau) + \alpha_2'(\tau)u_x(1,\tau)]\,d\tau.$$

By (15)–(17), (19), (46), (22), (27), the Young inequality, and the Gronwall lemma, the solution $u(x,t)$ to the boundary value problem (50), (2), (51), (52) satisfies the required a priori estimate

$$\|u\|_V \leqslant M_3,$$

where the constant M_3 is determined by the functions $a(x,t)$, $c(x,t)$, $\alpha_i(t)$, $\beta_i(t)$, $i = 1,2$, and the number T. This estimate implies the solvability of the problem (54), (2), (51), (52) in the space V for $\lambda = 1$. The function

$$u(x,t) = w_1(x,t) + \overline{b}_1(x,t,1)w_t(0,t) + \overline{b}(x,t,1)w_{1x}(1,t).$$

is the required solution to Problem III. The uniqueness of a solution is obvious. □

We set

$$r_1 = \max_{(x,t)\in\overline{Q}} \left[\int\limits_\Omega R_{xx}^2(x,y,t)\,dy\right], \quad r_2 = \max_{(x,t)\in\overline{Q}} \left[\int\limits_\Omega R^2(x,y,t)\,dy\right],$$

$$r_3 = \max_{(x,t)\in\overline{Q}} \left[\int\limits_\Omega R_0^2(x,y,t)\,dy\right], \quad r_4 = \max_{(x,t)\in\overline{Q}} \left[\int\limits_\Omega R_{0xx}^2(x,y,t)\,dy\right].$$

Theorem 4. *Suppose that* (9), (15), (16), (47)–(49), *and the following condition hold*:

$$\exists\,\gamma_0(x) \in \left(0, \frac{\sqrt{5}}{2}\right) : \; r_1(1+r_3)\left(4 + \frac{1}{4\gamma_0^2}\right) + 2r_2r_4\left(6 + \frac{1}{\gamma_0^2}\right) < \frac{1}{4}, \quad r_2\left(4 + \frac{1}{\gamma_0^2}\right) < \frac{1}{2}. \quad (54)$$

Then there exists a unique solution $u(x,t)$ in V to Problem II.

The proof of this theorem is similar to that of Theorem 2 differs by only the choice of δ_1–δ_6 ($\delta_1 = \delta_3 = \delta_4 = = \delta_6 = \frac{1}{\sqrt{2}}$, $\delta_2 = \sqrt{2}\gamma_0$, $\delta_5 = \gamma_0$).

4 Solvability of Problem III

The scheme of the proof of Problem III is the same as that for Problems I and II. We set

$$\Delta_3(t) = \beta_1(t) + \beta_2(t) - \alpha_1(t) - \alpha_2(t) - 1.$$

Assume that

$$\Delta_3(t) \neq 0, \quad t \in [0,T]. \quad (55)$$

We set

$$\varphi_0(t)=\frac{1}{\Delta_3(t)},\quad \psi_0(t)=-\frac{1}{\Delta_3(t)},\quad \varphi_1(t)=\frac{\beta_1(t)+\beta_2(t)-1}{\Delta_3(t)},\quad \psi_1(t)=-\frac{\alpha_1(t)+\alpha_2(t)}{\Delta_3(t)},$$
$$S_1(x,t)=\varphi_0(t)K_1(x)+\psi_0(t)K_2(x),\quad S_2(x,t)=\varphi_1(t)K_1(x)+\psi_1(t)K_2(x),$$
$$S(x,y,t)=xS_1(y,t)+S_2(y,t),\quad \widetilde{z}_1(t)=1-\int\limits_\Omega xS_1(x,t)\,dx,\quad \widetilde{z}_2(t)=-\int\limits_\Omega S_1(x,t)\,dx,$$
$$\widetilde{s}_1(t)=-\int\limits_\Omega xS_2(x,t)\,dx,\quad \widetilde{s}_2(t)=1-\int\limits_\Omega S_2(x,t)\,dx,\quad \Delta_{31}(t)=\widetilde{z}_1(t)\widetilde{s}_2(t)-\widetilde{z}_2(t)\widetilde{s}_(t),$$
$$S_0(x.y,t)=\frac{1}{\Delta_{31}(t)}\{[x\widetilde{s}_2(t)-\widetilde{s}_1(t)]S_1(y,t)-[x\widetilde{z}_2(t)-\widetilde{z}_1(t)]S_2(y,t)\}.$$

We introduce the operator B by the formula

$$(Bu)(x,t)=u(x,t)-\int\limits_\Omega S(x,y,t)u(y,t)\,dy.$$

In the case

$$\Delta_{31}(t)\neq 0,\quad t\in[0,T],\tag{56}$$

this operator is invertible and B^{-1} is defined by the equality

$$(B^{-1}v)(x,t)=v(x,t)+\int\limits_\Omega S_0(x,y,t)v(y,t)\,dy.$$

Theorem 5. *Suppose that* (14)–(19), (55), *and* (56) *hold. Then there exists a unique solution $u(x,t)$ in V to Problem* III.

We set

$$\widetilde{k}_1=\max_{(x,t)\in\overline{Q}}\left[\int\limits_\Omega S_{xx}^2(x,y,t)\,dy\right],\quad \widetilde{k}_2=\max_{(x,t)\in\overline{Q}}\left[\int\limits_\Omega S^2(x,y,t)\,dy\right],$$
$$\widetilde{k}_3=\max_{(x,t)\in\overline{Q}}\left[\int\limits_\Omega S_0^2(x,y,t)\,dy\right],\quad \widetilde{k}_4=\max_{(x,t)\in\overline{Q}}\left[\int\limits_\Omega S_{0xx}^2(x,y,t)\,dy\right].$$

Theorem 6. *Suppose that* (15)–(17), (19), (55), (56), *and the following condition hold*:

$$\exists\,\gamma_0\in\left(0,\frac{\sqrt{5}}{2}\right):\ \widetilde{k}_1(1+\widetilde{k}_2)\left(4+\frac{1}{4\gamma_0^2}\right)+2\widetilde{k}_2\widetilde{k}_4\left(6+\frac{1}{\gamma_0^2}\right)<\frac{1}{4},\quad \widetilde{k}_2\left(4+\frac{1}{\gamma_0^2}\right)<\frac{1}{2}.\tag{57}$$

Then there exists a unique solution $u(x,t)$ in V to Problem III.

The proof of Theorems 5 and 6 is similar to that of Theorems 1 and 3, 2 and 4 respectively. The main a priori estimate is obtained by analyzing an equality of the form (53).

Remarks. 1. Using the parameter continuation method and estimates obtained in the proof of Theorems 1–6, it is easy to establish the solvability in V of Problems I–III for loaded equations

$$Lu=f(x,t)+G(x,t,u)+F(x,t,\overline{u}(t)),$$

where $G(x,t,u)$ is the sum of integrals over Ω of the functions $u(x,t)$, $u_t(x,t)$, $u_x(x,t)$, $u_{xt}(x,t)$, $u_{xx}(x,t)$, $u_{xxt}(x,t)$ (with weights) and $F(x,t,\overline{u})$ that is a linear form of traces of the function

$u(x,t)$ and its derivatives at $x = 0$ and $x = 1$. Further, the operator L in these equations can be replaced with a more general operator of the form

$$u_{tt} - A(x,t)u_{xxt} - a_1(x,t)u_{xx} + a_2(x,t)u_x + b(x,t)u_t + c(x,t)u$$

such that $A(x,t) \geqslant a_0 > 0$ for $(x,t) \in \overline{Q}$, under some natural smoothness conditions.

2. From the proof of Theorems 2, 4, and 6 it immediately follows that the conditions (29), (54), and (57) can be slightly changed due to some other choice of the parameters δ_1–δ_6.

3. The conditions (29), (54), and (57) are smallness conditions. It is obvious that the set of data of Problems I–III satisfying these conditions is not empty.

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