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OPTIMUM DESIGN OF STRUCTURALLY NONHOMOGENEOUS MATERIALS AND CONSTRUCTIONS WITH REQUIRED PROPERTIES

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Based on mathematical modeling and computer simulation, the authors have investigated the problem of optimum design of composite materials and constructions with the required properties of plane and curvilinear symmetries in a variational statement. They have also performed a numerical analysis of their structure. Using the theory of continuation in parameter and structural analysis of differential equations describing the change in the properties of composite constructions, the authors have developed an efficient procedure of nonlocal optimization of the structure of such constructions with the aim of imparting necessary properties to them.

Keywords: *optimum design, nonhomogeneous structures, composite materials, variational statement, Pontryagin maximum principle.*

Widespread use of composite constructions on various fields of physics, technology, and instrument manufacture and the commonness of their mathematical description make it necessary to create a unified approach to investigation of the maximum capabilities of nonhomogeneous structures with required properties [1–4]. This calls for efficient methods to search for the global extremum of the corresponding effectiveness functionals in variational statements of problems of synthesis of the above materials.

Modern constructions must meet a broad set of requirements related to limitations on weight, cost, strength, rigidity, reliability, and resilience to the impact of destabilizing factors of various physical nature. Therefore, the problem of designing constructions with an optimum structure has received much attention to. Such constructions include shell constructions, especially ones with a layer structure, in which there is a combination of high bearing capacity and small mass. The layer structure of such constructions also ensures necessary heat-insulating and sound-proof and vibration-proof properties in them. This is relevant for constructions of space-rocket, aviation, and shipbuilding equipment. Composite materials of high strength and rigidity, whose mechanical and physical characteristics may vary within wide limits, can be utilized for creation of highly efficient multifunctional layer constructions with prescribed parameters. Raising the efficiency of the constructions is inextricably associated with improving approaches to their optimum design and calculation. The most universal is the variational approach making it possible to reduce the problem of optimum design of constructions to a problem of optimum control over the parameters of special systems.

To create effective necessary conditions for optimization of structural parameters, it is necessary to change the notion of control parameters′ closeness in investigating the optimum solution. In assessing closeness in the space of controls, it is only the control parameters that are in the small vicinity of an optimum solution that are accepted for comparison. However, the measure of closeness can be transferred from the space of controls to the space of states. In this case it is only phase trajectories that are close to the optimum phase trajectory that are accepted for comparison. Whereas in the space of controls the closeness of phase trajectories is determined by the closeness of controls, now the closeness of phase trajectories is determined independently. In the phase space of states, two controls are considered to be close if the phase trajectories produced by them are close. Consideration of the entire set U instead of a certain small vicinity of the optimum solution leads to a substantial enhancement of the necessary optimality conditions. The discreteness of the control region U is no longer an

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obstacle, since variations of the control parameters can be arbitrary. This transition from the space of controls to the space of phase states leads to the Pontryagin maximum principle [1, 5]. Therefore, in a variational statement, methods based on the indicated principle are some of the most efficient methods to construct optimum solutions in optimization problems in which the models are described by the systems of differential equations. However, methods based on the Pontryagin maximum principle enable one to construct just locally optimum solutions, i.e., are only local in character.

In accordance with this, it becomes necessary to develop novel efficient methods for optimum synthesis of composite materials with required properties and for numerical analysis of their structure that would be free of the above-said shortcomings. In a variational statement, this problem is associated with the development of methods for nonlocal optimum synthesis of materials that enable one to construct globally optimum solutions or solutions close to them to some extent [1–3].

The problem of investigating maximum capabilities of layer nonhomogeneous structures, as far as the required set of properties under wave actions on them are concerned, will be considered for the case of oblique incidence of electromagnetic waves on a plane-parallel layer structure. The propagation of electromagnetic waves inside the layers of this structure will be described by the system of Maxwell equations

$$\begin{gathered} \operatorname{rot}\mathbf{H}_s - \frac{\varepsilon_s}{c}\frac{\partial \mathbf{E}_s}{\partial t} = 0, \\ \operatorname{rot}\mathbf{E}_s + \frac{\mu_s}{c}\frac{\partial \mathbf{E}_s}{\partial t} = 0, \\ b_{s-1} \le z \le b_s, \quad s = 1, \ldots, N . \end{gathered} \tag{1}$$

where $\mathbf{H}_s$ and $\mathbf{E}_s$ are the vectors of the electric and magnetic electromagnetic field strengths in the sth layer, ε_s and μ_s are the permittivity and the permeability of the sth layer, b_s are the coordinates of the boundaries of layers with different physical properties, and N is the number of layers.

On the layer boundaries will hold the conditions of conjugation of solutions, which lie in the continuity of the normal and tangential components of the vectors of electric and magnetic strengths:

$$E_{s,\tau}\Big|_{z=b_{s-1}} = E_{s\text{-}1,\tau}\Big|_{z=b_{s-1}}, \quad H_{s,\tau}\Big|_{z=b_{s-1}} = H_{s\text{-}1,\tau}\Big|_{z=b_{s-1}} . \tag{2}$$

The boundary conditions reflect the interrelationship of the incident and reflected waves on the exterior and interior surfaces of the construction. As the index of solution efficiency in the variational statement under study, we have selected a standard measure of closeness of the energy transmission factor $T(\omega)$ to the required dependence $\tilde{T}(\omega)$ in the prescribed frequency range $[\omega_{\min}, \omega_{\max}]$:

$$J = \int_{\omega_{\min}}^{\omega_{\max}} \left[T(\omega) - \tilde{T}(\omega)\right]^2 d\omega \Rightarrow \min . \tag{3}$$

Here

$$T(\omega) = \Pi_z^{\text{out}} / \Pi_z^{\text{in}} , \tag{4}$$

where Π_z^{out} and Π_z^{in} are the projections of the Poynting vector onto the z axis in the transmitted and incident waves respectively.

The employment of the procedure of optimum synthesis, which is based on the L. S Pontryagin maximum principle and involves spiky variation of the allowable solutions, enables one to synthesize composite structures for which the functional dependences of the energy characteristics on frequency are more effective throughout the frequency range than the corresponding dependences of the energy coefficients constructed using the existing approaches.

For the case of oblique incidence of electromagnetic waves on a composite construction, the necessary optimality conditions can be formulated in the following form. We introduce a system of scalar functions R_k $(1 \le k \le N)$, which are an analog of Hamilton functions in the variational statement of problems of optimum synthesis of materials on exposure to electromagnetic waves:

$$R_k(;\, \varepsilon)\Big|_z = R_k^{\perp}(;\, \varepsilon)\Big|_z + R_k^{\parallel}(;\, \varepsilon)\Big|_z , \quad 1 \le k \le N ; \tag{5}$$

$$R_k^{\perp}(;\,\varepsilon)\Big|_z = -\int\limits_{\omega_{\min}}^{\omega_{\max}} \mathrm{Re}\left[\alpha_k \left(\frac{\varepsilon}{\varepsilon_k} - \frac{\varepsilon_0\mu_0 \sin^2\vartheta_0}{\varepsilon_k \mu(\varepsilon)}\right) f_k(z,\omega)\,\frac{\partial \psi_k(z,\omega)}{\partial z}\right] d\omega$$

$$+ \int\limits_{\omega_{\min}}^{\omega_{\max}} \mathrm{Re}\left[\alpha_k\,\frac{\mu(\varepsilon)}{\mu_k}\,\psi_k(z,\omega)\,\frac{\partial f_k(z,\omega)}{\partial z}\right] d\omega\,,$$

$$R_k^{\parallel}(;\,\varepsilon)\Big|_z = -\int\limits_{\omega_{\min}}^{\omega_{\max}} \mathrm{Re}\left[\alpha_k \left(\frac{\mu(\varepsilon)}{\mu_k} - \frac{\varepsilon_0\mu_0 \sin^2\vartheta_0}{\varepsilon \mu_k}\right) g_k(z,\omega)\,\frac{\partial p_k(z,\omega)}{\partial z}\right] d\omega \qquad (6)$$

$$+ \int\limits_{\omega_{\min}}^{\omega_{\max}} \mathrm{Re}\left[\frac{\varepsilon}{\varepsilon_k}\,p_k(z,\omega)\,\frac{\partial g_k(z,\omega)}{\partial z}\right] d\omega\,,$$

$$\alpha_k = \left(1 - \frac{\varepsilon_0\mu_0}{\varepsilon_k\mu_k}\sin^2\vartheta_0\right)^{-1}.$$

Here $R_k^{\perp}(;\,\varepsilon)\Big|_z$ and $R_k^{\parallel}(;\,\varepsilon)\Big|_z$ are the vertical and horizontal components of the electromagnetic wave, the functions $f_k(z,\omega)$ and $g_k(z,\omega)$ have the meaning of complex amplitudes for the components of the electromagnetic wave with vertical and horizontal components, the functions $\psi_k(z,\omega)$ and $p_k(z,\omega)$ are the complex solutions of the corresponding conjugate boundary-value problem [1–4], ε_k and μ_k are the permittivity and the permeability of the kth layer of the construction, ε is the variable parameter from the allowable set of permittivities of materials from the allowable set Λ used during the optimum synthesis of a composite construction, $\mu(\varepsilon)$ is the permeability of the material with a permittivity ε, and ϑ_0 is the angle of incidence of an electromagnetic wave on a composite construction. Then the optimum structure of the composite construction, which is determined by the optimum number of layers N^*, the optimum parameters of permittivity and permeability inside the sth layer of the composite construction ε_s^* and μ_s^* ($s = 1, \ldots, N^*$), and the optimum coordinates of the boundaries of the layers b_s^* ($s = 1, \ldots, N^*$), satisfies the relations of the maximum principle

$$R_k(*;\,\varepsilon_k^*) = \max_{\varepsilon\in\Lambda} R(*;\,\varepsilon)\,,$$
$$b_{k-1}^* \le z \le b_k^*, \quad (k = 1, \ldots, N^*)\,. \qquad (7)$$

Despite the fact that optimum-synthesis methods, which are based on optimality conditions related to nonlocal parametric variations like the Pontryagin maximum principle, enable one to construct effective solutions, nonetheless, the resulting solutions are locally optimum ones. Therefore, the employment of the procedure relying on the Pontryagin maximum method makes it impossible to investigate maximum capabilities of layer nonhomogeneous composite structures as far as a required set of their properties is concerned.

The absence of the efficient methods to investigate maximum capabilities of layer nonhomogeneous constructions makes it impossible, on the one hand, to design in an optimum manner such constructions with characteristics as close as possible to the required ones, and on the other, gives no way of assessing the extent to which the characteristics of multilayer constructions functioning in various fields of physics and technology differ from maximum attainable ones. One of the most efficient approaches to solution of this problem is the approach based on the theory of multiple-valued mappings [1–3, 6] and on the established property of internal structural symmetry of the optimum composite constructions. This approach enables one to investigate maximum capabilities of layer nonhomogeneous compositions as far as the prescribed set of their properties is concerned.

From the structural investigation of necessary optimality conditions related to nonlocal variations of control parameters, we have established the existence of internal symmetry in the interrelationship of parameters determining the physical and geometric structures of an optimum construction. The existence of such internal symmetry in the problem of optimum synthesis of layer nonhomogeneous constructions may suggest that structures implementing their maximum capabilities will only be grouped within a narrow compact set Q. The internal order or internal symmetry in the interrelationship of the optimum parameters may lead to the fact that structures implementing their maximum capabilities will satisfy additional relations. Establishing such relations makes it possible to substantially reduce the dimensions of the problem, i.e., it may turn out that structures implementing their maximum capabilities additionally satisfy a certain system of m equations $M_j(u^*) = 0$. The set of solutions of this system is precisely the sought compact set

$$Q = \{u : Q_j(u) = 0, \quad j = 1, \ldots, m\} .$$

Constructing the indicated system of equations enables us to completely solve the synthesis problem in a number of cases. Here the main difficulty is to develop a procedure for analytical description of the boundaries of a singled-out compact set. Therefore, it is of considerable interest to single out problems of synthesis of layer systems in which structures implementing their maximum capabilities, as far as the control over wave-field parameters is concerned, are of internal symmetry. With the investigation of the possibility of singling out the narrow compact set Q that contains the entire combination of variants implementing maximum capabilities, there is associated a qualitatively new method of contraction of the set of allowable variants of the structures and development, on this basis, of efficient synthesis methods. For a certain range of wave problems of synthesis, it has been shown that there is internal symmetry in the interrelationship of parameters in layer nonhomogeneous compositions implementing their maximum capabilities. This makes it possible to substantially reduce their dimensions [1–3, 6]. In such problems, which will be called reference problems in the subsequent discussion, the combination of all the variants of multilayer interference coatings implementing their maximum capabilities, as far as the control over wave-field parameters is concerned, turns out to belong to the narrow compact set.

We have developed the procedure of analytical description of the boundaries of the singled-out compact set. For the reference optimum-synthesis problems, we can efficiently single out a combination of all the variants of layer structures whose parameters deliver the global minimum to the quality functional characterizing the closeness of the functional characteristics to the required ones. Nonetheless, there is a wide range of wave synthesis problems in various fields of physics and technology for which it seems impossible to analytically describe the boundaries of compact sets containing the entire combination of optimum solutions. However, these problems can be related, in a way, to the reference problems of synthesis. For example, in certain techniques of introduction of a parameter into the model, they can be found in one parametric family, such that to one parametric value there corresponds the initial problem of synthesis, and to another, the reference problem. This brings about the problem of how one, knowing the optimum solutions of the reference synthesis problem, can develop efficient methods to investigate maximum capabilities for a wide range of wave problems in various fields of physics and technology.

Let us place the initial problem of optimum synthesis in a parametric family of optimum-synthesis problems that is dependent on the real parameter δ ($\delta_0 \le \delta \le \delta_1$). Here to the parametric value $\delta = \delta_0$ corresponds the reference synthesis problem for which we can single out efficiently the entire combination of optimum solutions implementing the maximum capabilities as far as the prescribed set of properties $U^*(\delta_0)$ is concerned. To the parametric value $\delta = \delta_1$ there corresponds the initial problem of optimum synthesis. The set of globally optimum solutions in the initial synthesis problem will be denoted by U^*; here $U^* = U^*(\delta_1)$. A procedure of continuation of solution in parameter has been developed on the basis of the theory of multiple-valued mappings [7]. This procedure enables one to efficiently continue the set of globally optimum solutions of the reference problem of synthesis in parameter. The developed procedure of investigation of the maximum capabilities, which is based on the theory of multiple-valued mappings and on the methods of continuation in parameter, makes it possible to efficiently design layer nonhomogeneous structures with complex characteristics under wave actions.

Let us consider the employment of the developed procedure of optimum synthesis for designing a multilayer filter ensuring high transmission of the wave energy in a narrow spectral region and high reflection in the remaining region of the considered wavelength range [λ_{min}, λ_{max}] for which the ratio of the maximum wavelength to the minimum wavelength is equal to $\lambda_{max}/\lambda_{min} = 3.4$. The total thickness of the layer structure is $l = 9.78\lambda_{min}$.The allowable set consists of two materials with refractive indices $n_{min} = 1.46$ and $n_{max} = 2.0$. The refractive indices of the media surrounding the multilayer structure are equal respectively to $n_0 = 1$ and $n_{N+1} = 1$. It is necessary to ensure high transmission of the electromagnetic wave in the narrow spectral region [λ_{in}, λ_f] $\subset$ [λ_{min}, λ_{max}] and high reflection in the remaining part of the considered spectral range, where $\lambda_{in} = 1.2\lambda_{min}$ and $\lambda_f = 1.36\lambda_{min}$. The required dependence of the energy transmission factor on the wavelength in the example in question is of the form

$$\tilde{T}(\lambda) = \begin{cases} 1, & \lambda \in [\lambda_{in}, \lambda_f] , \\ 0, & \lambda \in [\lambda_{in}, \lambda_f] . \end{cases}$$

The dependence of the energy transmission factor on the wavelength for the resulting coating constructed according to the procedure based on the theory of multiple-valued mappings and the continuation in parameter is given in Fig. 1. The

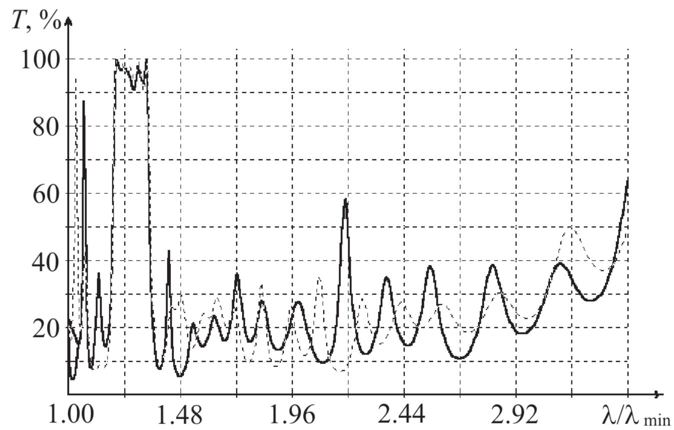

Fig. 1. Energy transmission factor of the optimum multilayer structure vs. wavelength: solid curve, calculation according to the proposed procedure; dashed curve, calculation by the method of the Pontryagin maximum principle.

number of layers in the resulting multilayer structure is $N^* = 34$. The same figure gives the dependence $T(\lambda)$ for the coating resulting from the employment of the procedure based on the Pontryagin maximum principle.

Thus, the employment of the developed procedure enables one to design effective layer nonhomogeneous structures with complex characteristics under wave actions that exceed in characteristics nonhomogeneous structures synthesized using the existing methods.

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NOTATION

T, energy transmission factor; λ, wavelength; λ_{in} and λ_f, initial and final wavelengths; λ_{min} and λ_{max}, minimum and maximum wavelengths.

REFERENCES

1. E. L. Gusev, *Mathematical Methods of Synthesis of Layer Structures* [in Russian], Nauka, Novosibirsk (1993).
2. E. L. Gusev, Qualitative regularities of the interrelationship of parameters in optimum structures in problems of synthesis of nonhomogeneous structures from a discrete set of materials under wave actions, *Dokl. Ross. Akad. Nauk*, **346**, No. 3, 324–326 (1996).
3. E. L. Gusev, Optimal synthesis methodology of nonhomogeneous structures under the influence of electromagnetic waves, *Int. J. Appl. Electromagn. Mech*., No. 10, 405–416 (1999).
4. V. N. Bakulin, E. L. Gusev, and V. G. Markov, Methods of optimum design and calculation of composite constructions, in: *Optimum Design of Constructions from Composite and Traditional Materials* [in Russian], Vol. 1, Nauka, Moscow (2008).
5. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *Mathematical Theory of Optimum Processes* [in Russian], Nauka, Moscow (1983).
6. E. L. Gusev, The property of internal symmetry in the interrelationship of parameters in optimum structures in problems of optimum synthesis of nonhomogeneous structures under wave actions, *Proc. Inst. Conf. "Symmetry in Natural Science,"* Krasnoyarsk (1998), pp. 47–48.
7. O. A. Liskovets, Ill-posed problems with a closed irreversible operator, *Differents. Uravn*., No. 4, 636–646 (1967).