ISSN 0012-2661, Differential Equations, 2015, Vol. 51, No. 3, pp. 362–375. © Pleiades Publishing, Ltd., 2015.
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PARTIAL DIFFERENTIAL EQUATIONS

Solvability of a Boundary Value Problem for a Pseudoparabolic Equation with Nonlocal Integral Conditions

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Received October 26, 2013

Abstract— By using the method of continuation with respect to a parameter and a priori estimates, we prove the unique regular solvability of a boundary value problem with nonlocal integral boundary conditions for a one-dimensional pseudoparabolic equation.

DOI: 10.1134/S0012266115030076

1. INTRODUCTION

In the present paper, we study the solvability of spatially nonlocal boundary value problems for one-dimensional linear pseudoparabolic equations with integral boundary conditions. Our methods are based on the passage from a problem for a "good" equation with "bad" boundary conditions to a problem with "good" boundary conditions but for a "bad" equation, known as a loaded equation [1, pp. 13, 27], and on the proof of the solvability of the resulting problem by the method of continuation with respect to a parameter and a priori estimates. Earlier, similar methods in a close situation were efficiently used in [2–4].

2. STATEMENT OF THE PROBLEM

Let Ω be the interval $(0,1)$ of the axis Ox, and let Q be the rectangle $\Omega \times (0,T)$, $0 < T < +\infty$. In the domain Q, consider the equation

$$u_t - a(x,t)u_{xx} + c(x,t)u - u_{xxt} = f(x,t), \qquad (x,t) \in Q, \tag{1}$$

with the nonlocal integral conditions

$$\int\limits_0^1 H_i(x,t)u(x,t)\,dx = 0, \qquad i = 1,2, \tag{2}$$

where $a(x,t)$, $c(x,t)$, $f(x,t)$, and $H_i(x,t)$ $(i = 1,2)$ are given functions defined for $x \in \overline{\Omega} = [0,1]$ and $t \in [0,T]$.

Boundary value problem. Find a function $u(x,t)$ that is a solution of Eq. (1) in the rectangle Q and satisfies condition (2) and the initial condition

$$u(x,0) = 0, \qquad x \in \Omega. \tag{3}$$

Note that the solvability of the initial–boundary value problem for the heat equation

$$u_t - a(x,t)u_{xx} + c(x,t)u = f(x,t) \tag{4}$$

was studied in [2] by the method of regularization and continuation with respect to a parameter.

In the present paper, we prove the solvability of boundary value problems with the nonlocal boundary conditions (2) by the methods used in the study of Eq. (4). In the case of local boundary conditions, solvability theorems for Eq. (1), which is called a pseudoparabolic equation, or the Aller equation, were proved in [5, p. 140; 6, p. 29].

Note also that problems for the heat equation (4) with constant coefficients and with general nonlocal Samarskii conditions with constant coefficients were studied in [7] by the Fourier method.

A preliminary study of boundary value problems with nonlocal Samarskii boundary conditions with variable coefficients was carried out in [8–10].

3. MAIN THEOREM

If we multiply the original equation (1) by $H_i(x,t)$ and integrate the resulting relation over the domain Ω, then, by using condition (2), we obtain the system of two equations

$$\begin{aligned} &H_i(0,t)u_{xt}(0,t) - H_i(1,t)u_{xt}(1,t) \\ &= -H_i(0,t)a(0,t)u_x(0,t) + H_i(1,t)a(1,t)u_x(1,t) + H_{ix}(0,t)u_t(0,t) - H_{ix}(1,t)u_t(1,t) \\ &\quad + (H_i\cdot a)_x(0,t)u(0,t) - (H_i\cdot a)_x(1,t)u(1,t) + \int\limits_\Omega [(H_ia)_{xx}(x,t) - (H_ic)(x,t) + H_{it}(x,t)]u(x,t)\,dx \\ &\quad + \int\limits_\Omega H_{ixx}(x,t)u_t(x,t)\,dx + \int\limits_\Omega H_i(x,t)f(x,t)\,dx, \qquad i = 1,2. \end{aligned} \tag{5}$$

Suppose that the determinant of the matrix A on the left-hand side in system (5) is nonzero for $t \in [0,T]$,

$$|A| = -H_1(0,t)H_2(1,t) + H_1(1,t)H_2(0,t) \neq 0.$$

Then, instead of the nonlocal boundary conditions (5), we obtain conditions of the form

$$\begin{aligned} u_{xt}(0,t) &= \alpha_1(t)u_x(0,t) + \alpha_2(t)u_x(1,t) + \alpha_3(t)u_t(0,t) + \alpha_4(t)u_t(1,t) \\ &\quad + \alpha_5(t)u(0,t) + \alpha_6(t)u(1,t) + \int_0^1 K_1(x,t)u(x,t)\,dx + \int_0^1 N_1(x,t)u_t(x,t)\,dx, \end{aligned} \tag{6}$$

$$\begin{aligned} u_{xt}(1,t) &= \beta_1(t)u_x(0,t) + \beta_2(t)u_x(1,t) + \beta_3(t)u_t(0,t) + \beta_4(t)u_t(1,t) \\ &\quad + \beta_5(t)u(0,t) + \beta_6(t)u(1,t) + \int_0^1 K_2(x,t)u(x,t)\,dx + \int_0^1 N_2(x,t)u_t(x,t)\,dx, \end{aligned} \tag{7}$$

where $\alpha_k(t)$, $\beta_k(t)$ $(k = 1,\dots,6)$, $K_l(x,t)$, and $N_l(x,t)$ $(l = 1,2)$ are given functions defined for $x \in \overline{\Omega}$ and $t \in [0,T]$. Without loss of generality, we assume that

$$\int\limits_\Omega H_i(x,t)f(x,t)\,dx = 0, \qquad i = 1,2.$$

Boundary value problem 1. Find a function $u(x,t)$ that is a solution of Eq. (1) in the rectangle Q and satisfies the nonlocal boundary conditions (6) and (7) and the initial condition (3).

We introduce the following notation:

$$\varphi_4(t) = \frac{1}{2}\int_0^1 x^2 N_2(x,t)\,dx, \qquad \psi_4(t) = \int_0^1 x N_2(x,t)\,dx.$$

Note that the numbering of the functions φ_4 and ψ_4 is introduced for convenience and will be explained below.

Let the norm on the space

$$V = \{v(x,t):\ v(x,t), v_t(x,t), v_{xx}(x,t), v_{xxt}(x,t) \in L_2(Q),\ v_x(x,t) \in L_\infty(0,T;W_2^1(0,1))\}$$

be defined by the formula

$$\|v\|_V = \|v\|_{W_2^{2,1}(Q)} + \|v_{xxt}\|_{L_2(Q)} + \|v_x\|_{L_\infty(0,T;W_2^1(\Omega))}.$$

Theorem. *Let the following conditions be satisfied*:

$$\begin{aligned}
&a(x,t) \in C^1(\overline{Q}), \qquad c(x,t) \in C^1(\overline{Q}), \qquad a(x,t) \ge \overline{a}_0 > 0,\\
&c(x,t) \ge \overline{c}_0 > 0, \qquad (x,t) \in \overline{Q};\\
&\alpha_i(t) \in C^1([0,T]), \qquad \beta_i(t) \in C^1([0,T]), \qquad i = 1,\dots,6; \qquad (8)\\
&K_p(x,t) \in C^1(\overline{Q}), \qquad N_p(x,t) \in C^1(\overline{Q}), \qquad p = 1,2;\\
&\alpha_4(t) + \beta_4(t) \ne 2, \quad \alpha_4(t)[1 + \psi_4(t) - 4\varphi_4(t)] + \beta_4(t)[1 - \psi_4(t)] \ne 2 - 2\psi_4(t) - 4\varphi_4(t), \quad t \in [0,T];\\
&q(\xi_1,\xi_2,t) \equiv \alpha_3(t)\xi_1^2 + (\alpha_4(t) - \beta_3(t))\xi_1\xi_2 - \beta_4(t)\xi_2^2 \ge 0, \qquad t \in [0,T], \qquad (\xi_1,\xi_2) \in \mathbb{R}^2, \qquad (9)\\
&f(x,t) \in L_2(Q). \qquad (10)
\end{aligned}$$

Then there exists a unique solution $u(x,t) \in V$ of the boundary value problem 1 *in the rectangle Q.*

Remark 1. Obviously, the conditions $\alpha_4(t) + \beta_4(t) \ne 2$ and

$$\alpha_4(t)[1 + \psi_4(t) - 4\varphi_4(t)] + \beta_4(t)[1 - \psi_4(t)] \ne 2 - 2\psi_4(t) - 4\varphi_4(t)$$

are satisfied if the given functions $\alpha_4(t)$, $\beta_4(t)$, and $N_2(x,t)$ are small in absolute value.

4. SOLVABILITY OF THE AUXILIARY BOUNDARY VALUE PROBLEM

Consider an auxiliary boundary value problem for Eq. (1).

Boundary value problem 2. Find a function $u(x,t)$ that is a solution of Eq. (1) in the rectangle Q and satisfies the nonlocal boundary conditions

$$\begin{aligned}
u_{xt}(0,t) = \alpha_1(t)u_x(0,t) + \alpha_2(t)u_x(1,t) + \alpha_3(t)u_t(0,t)&\\
+ \alpha_4(t)u_t(1,t) + \alpha_5(t)u(0,t) + \alpha_6(t)u(1,t),& \qquad (11)\\
u_{xt}(1,t) = \beta_1(t)u_x(0,t) + \beta_2(t)u_x(1,t) + \beta_3(t)u_t(0,t)&\\
+ \beta_4(t)u_t(1,t) + \beta_5(t)u(0,t) + \beta_6(t)u(1,t)& \qquad (12)
\end{aligned}$$

and the initial condition (3).

Lemma. *Let conditions* (9) *and* (10) *be satisfied, and let*

$$\begin{aligned}
&a(x,t) \in C^1(\overline{Q}), \qquad c(x,t) \in C^1(\overline{Q}), \qquad a(x,t) \ge \overline{a}_0 > 0,\\
&c(x,t) \ge \overline{c}_0 > 0, \qquad (x,t) \in \overline{Q};\\
&\alpha_i(t) \in C^1([0,T]), \qquad \beta_i(t) \in C^1([0,T]), \qquad i = 1,\dots,6;\\
&\alpha_4(t) + \beta_4(t) \ne 2, \qquad t \in [0,T].
\end{aligned} \qquad (13)$$

Then there exists a unique solution $u(x,t) \in V$ of the boundary value problem 2 *in the rectangle Q.*

Proof. For $(x,t) \in \overline{Q}$ and $\lambda \in [0,1]$, we set

$$\gamma_k(x,t,\lambda) = \frac{\lambda x^2}{2}[\beta_k(t) - \alpha_k(t)] + \lambda x\alpha_k(t), \qquad k = 1,\dots,6,$$

and introduce the function

$$v(x,t,\lambda) = \int_0^t [\gamma_1 u_x(0,\tau) + \gamma_2 u_x(1,\tau) + \gamma_3 u_t(0,\tau) + \gamma_4 u_t(1,\tau) + \gamma_5 u(0,\tau) + \gamma_6 u(1,\tau)]\, d\tau.$$

Note that the function $v(x,t)$ satisfies the relations

$$\begin{aligned} v_{xt}(0,t) &= \lambda[\alpha_1(t)u_x(0,t) + \alpha_2(t)u_x(1,t) + \alpha_3(t)u_t(0,t) \\ &\quad + \alpha_4(t)u_t(1,t) + \alpha_5(t)u(0,t) + \alpha_6(t)u(1,t)], \\ v_{xt}(1,t) &= \lambda[\beta_1(t)u_x(0,t) + \beta_2(t)u_x(1,t) + \beta_3(t)u_t(0,t) + \beta_4(t)u_t(1,t) + \beta_5(t)u(0,t) + \beta_6(t)u(1,t)]. \end{aligned}$$

Consider the function $w(x,t) = u(x,t) - v(x,t)$, where $u(x,t)$ is a solution of the boundary value problem (1), (11), (12), (3). By substituting the function $u(x,t) = w(x,t) + v(x,t)$ into the original equation (1), we obtain

$$w_t - a(x,t)w_{xx} + c(x,t)w - w_{xxt} = f(x,t) - v_t + a(x,t)v_{xx} - c(x,t)v + v_{xxt}. \tag{14}$$

We express the function $v(x,t)$ via the newly introduced function $w(x,t)$ from the system of equations

$$\begin{gathered} w_x(0,t) = u_x(0,t) - \int_0^t \sum_{i=1}^6 \alpha_i(\tau)u_i(\tau)\, d\tau, \qquad w_x(1,t) = \int_0^t \sum_{i=1}^6 \beta_i(\tau)u_i(\tau)\, d\tau, \\ w_t(0,t) = u_t(0,t), \qquad w_t(1,t) = u_t(1,t) - \frac{1}{2}\sum_{i=1}^6 (\alpha_i(t) + \beta_i(t))u_i(t), \\ w(0,t) = u(0,t), \qquad w(1,t) = u(1,\tau) - \frac{1}{2}\int_0^t \sum_{i=1}^6 (\alpha_i(\tau) + \beta_i(\tau))u_i(\tau)\, d\tau, \end{gathered} \tag{15}$$

where $(u_1, u_2, u_3, u_4, u_5, u_6) = (u_x(0,t), u_x(1,t), u_t(0,t), u_t(1,t), u(0,t), u(1,t))$.

From the fourth equation in system (15), we find $u_t(1,t)$. If the condition $\alpha_4(t) + \beta_4(t) \neq 2$ is satisfied, then from (8), we obtain

$$\begin{aligned} u_t(1,t) = \frac{1}{2 - \alpha_4(t) - \beta_4(t)} &[2w_t(1,t) + (\alpha_1(t) + \beta_1(t))u_x(0,t) + (\alpha_2(t) + \beta_2(t))u_x(1,t) \\ &+ (\alpha_3(t) + \beta_3(t))u_t(0,t) + (\alpha_5(t) + \beta_5(t))u(0,t) + (\alpha_6(t) + \beta_6(t))u(1,t)]. \end{aligned} \tag{16}$$

By eliminating the function $u_t(1,t)$ given by (16) from system (15), we obtain the system of five equations

$$\begin{aligned} w_x(0,t) = u_x(0,t) - \frac{1}{2 - \alpha_4 - \beta_4} &\int_0^t \{2\alpha_4 w_\tau(1,\tau) + [(2 - \beta_4)\alpha_1 + \alpha_4\beta_1]u_x(0,\tau) \\ &+ [(2 - \beta_4)\alpha_2 + \alpha_4\beta_2]u_x(1,\tau) + [(2 - \beta_4)\alpha_3 + \alpha_4\beta_3]u_\tau(0,\tau) \\ &+ [(2 - \beta_4)\alpha_5 + \alpha_4\beta_5]u(0,\tau) + [(2 - \beta_4)\alpha_6 + \alpha_4\beta_6]u(1,\tau)\}\, d\tau, \\ w_x(1,t) = u_x(1,t) - \frac{1}{2 - \alpha_4 - \beta_4} &\int_0^t \{2\beta_4 w_\tau(1,\tau) + [(2 - \alpha_4)\beta_1 + \beta_4\alpha_1]u_x(0,\tau) \\ &+ [(2 - \alpha_4)\beta_2 + \beta_4\alpha_2]u_x(1,\tau) + [(2 - \alpha_4)\beta_3 + \beta_4\alpha_3]u_\tau(0,\tau) \\ &+ [(2 - \alpha_4)\beta_5 + \beta_4\alpha_5]u(0,\tau) + [(2 - \alpha_4)\beta_6 + \beta_4\alpha_6]u(1,\tau)\}\, d\tau, \\ w_t(0,t) = u_t(0,t), \quad w(0,t) = u(0,t),& \\ w(1,t) = u(1,t) - \frac{1}{2 - \alpha_4 - \beta_4} &\int_0^t \{(\alpha_4 + \beta_4)w_\tau(1,\tau) + (\alpha_1 + \beta_1)u_x(0,\tau) \\ &+ (\alpha_2 + \beta_2)u_x(1,\tau) + (\alpha_3 + \beta_3)u_\tau(0,\tau) + (\alpha_5 + \beta_5)u(0,\tau) + (\alpha_6 + \beta_6)u(1,\tau)\}\, d\tau. \end{aligned} \tag{17}$$

One can rewrite system (17) in the form

$$\begin{aligned}
&\vec{w}_0(t) = \vec{u}_0(t) - \frac{1}{2-\alpha_4-\beta_4}\left[\int\limits_0^t B\vec{u}_0\,d\tau - \int\limits_0^t \vec{c}w_\tau(1,\tau)\,d\tau\right],\\
&\vec{w}_0(t) = (w_x(0,t), w_x(1,t), w_t(0,t), w(0,t), w(1,t)) \equiv (w_1, w_2, w_3, w_5, w_6),\\
&\vec{u}_0(t) = (u_x(0,t), u_x(1,t), u_t(0,t), u(0,t), u(1,t)) \equiv (u_1, u_2, u_3, u_5, u_6),
\end{aligned} \tag{18}$$

where

$$B = \begin{pmatrix}
(2-\beta_4)\alpha_1+\alpha_4\beta_1 & (2-\beta_4)\alpha_2+\alpha_4\beta_2 & (2-\beta_4)\alpha_3+\alpha_4\beta_3 & (2-\beta_4)\alpha_5+\alpha_4\beta_5 & (2-\beta_4)\alpha_6+\alpha_4\beta_6\\
(2-\alpha_4)\beta_1+\beta_4\alpha_1 & (2-\alpha_4)\beta_2+\beta_4\alpha_2 & (2-\alpha_4)\beta_3+\beta_4\alpha_3 & (2-\alpha_4)\beta_5+\beta_4\alpha_5 & (2-\alpha_4)\beta_6+\beta_4\alpha_6\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
\alpha_1+\beta_1 & \alpha_2+\beta_2 & \alpha_3+\beta_3 & \alpha_5+\beta_5 & \alpha_6+\beta_6
\end{pmatrix},$$

and $\vec{c} = (2\alpha_4, 2\beta_4, 0, 0, \alpha_4+\beta_4)$.

The integral equation (18) is a Volterra integral equation of the second kind [11, p. 102; 12, p. 110] and has a unique solution, which has the form

$$\vec{u}_0(t) = \vec{w}_0(t) + \int\limits_0^t \vec{F}_0(\tau, w_\tau(1,\tau), \vec{w}_0)\,d\tau, \qquad \vec{F}_0 = (F_{01}, F_{02}, 0, 0, F_{05}).$$

By substituting the function $\vec{u}_0$ in (18) and $u_t(1,t)$ in (16) into system (17), we obtain

$$v(x,t) = \int\limits_0^t F(x,\tau,\lambda,\vec{w}(\tau))\,d\tau, \tag{19}$$

where $\vec{w}(t) = (w_x(0,t), w_x(1,t), w_t(0,t), w_t(1,t), w(0,t), w(1,t))$. Now, by substituting the function $v(x,t)$ in (19) into Eq. (14), we finally obtain

$$\begin{aligned}
&w_t - a(x,t)w_{xx} + c(x,t)w - w_{xxt}\\
&\quad= f(x,t) - F(x,t,\lambda,\vec{w}(t)) + a(x,t)\int\limits_0^t F_{xx}(x,\tau,\lambda,\vec{w}(\tau))\,d\tau - c(x,t)\int\limits_0^t F(x,\tau,\lambda,\vec{w}(\tau))\,d\tau\\
&\qquad+ F_{xx}(x,t,\lambda,\vec{w}(t)) \equiv f(x,t) + \Phi(x,t,\lambda,\vec{w}(t)).
\end{aligned} \tag{20}$$

Consider the following auxiliary boundary value problem: find a function $w(x,t)$ that is a solution of Eq. (20) in the rectangle Q and satisfies the conditions

$$w(x,0) = 0, \qquad x \in \Omega, \tag{21}$$

$$w_{xt}(0,t) = w_{xt}(1,t) = 0, \qquad t \in (0,T). \tag{22}$$

Note that if $\lambda = 1$, then the boundary value problem (20)–(22) is equivalent to the original boundary value problem (1), (11), (12), (3).

Let us show that, under conditions (9), (10), and (13), this boundary value problem has a solution in the space V.

By Λ we denote the set of numbers λ in the interval $[0,1]$ for which the boundary value problem (20)–(22) has a solution $w(x,t)$ in the space V under conditions (9), (10), and (13). If we show that the set Λ is nonempty, is open and closed, then it coincides with the entire interval $[0,1]$ [13, p. 153].

If $\lambda = 0$, then, under conditions (9), (10), and (13), the boundary value problem (20)–(22) is solvable in the space V (see [4, p. 140; 6, p. 29]). Hence it follows that zero belongs to the set V; consequently, the set Λ is nonempty.

The fact that the set Λ is open and closed can be proved with the use of a priori estimates. Let us derive the corresponding a priori estimate.

Let $w(x,t)$ be a solution of the boundary value problem (20)–(22) in the space V. By using relation (19), we set

$$u(x,t) = w(x,t) + \int_0^t F(x,\tau,\lambda,\vec{w}(\tau))\,d\tau.$$

By using the inequality

$$\int_0^t [v^2(0,\tau) + v^2(1,\tau)]\,d\tau \le \delta \int_0^t\int_\Omega v_x^2(x,\tau)\,dx\,d\tau + C(\delta)\int_0^t\int_\Omega v^2(x,\tau)\,dx\,d\tau, \qquad (23)$$

where δ is an arbitrary positive number, one can readily show that the function $u(x,t)$ belongs to the space V. Note that the function $u(x,t)$ is a solution of the following initial–boundary value problem: find a function $u(x,t)$ that is a solution of Eq. (1) in the rectangle Q and satisfies the conditions

$$\begin{aligned} u_{xt}(0,t) &= \lambda[\alpha_1(t)u_x(0,t) + \alpha_2(t)u_x(1,t) + \alpha_3(t)u_t(0,t) \\ &\quad + \alpha_4(t)u_t(1,t) + \alpha_5(t)u(0,t) + \alpha_6(t)u(1,t)], \end{aligned} \qquad (24)$$

$$\begin{aligned} u_{xt}(1,t) &= \lambda[\beta_1(t)u_x(0,t) + \beta_2(t)u_x(1,t) \\ &\quad + \beta_3(t)u_t(0,t) + \beta_4(t)u_t(1,t) + \beta_5(t)u(0,t) + \beta_6(t)u(1,t)], \end{aligned} \qquad (25)$$

$$u(x,0) = 0. \qquad (26)$$

Consider the relation

$$\int_0^t\int_\Omega (u_\tau - au_{xx} + cu - u_{xx\tau})(u_\tau - u_{xx\tau})\,dx\,d\tau = \int_0^t\int_\Omega f(u_\tau - u_{xx\tau})\,dx\,d\tau;$$

by integrating by parts, we obtain the relation

$$\begin{aligned} &\int_0^t\int_\Omega [u_\tau^2 + 2u_{x\tau}^2 + u_{xx\tau}^2]\,dx\,d\tau + \frac{1}{2}\int_\Omega a(x,t)u_{xx}^2(x,t)\,dx + \frac{1}{2}\int_\Omega [a(x,t) + c(x,t)]u_x^2(x,t)\,dx \\ &\quad + \frac{1}{2}\int_\Omega c(x,t)u^2(x,t)\,dx + 2\int_0^t [u_{x\tau}(0,\tau)u_\tau(0,\tau) - u_{x\tau}(1,\tau)u_\tau(1,\tau)]\,d\tau \\ &= \int_0^t\int_\Omega \left\{\frac{1}{2}[a_\tau u_{xx}^2 + (a_\tau + c_\tau)u_x^2 + c_\tau u^2] - a_x u_x u_\tau - c_x u u_{x\tau}\right\}\,dx\,d\tau \\ &\quad + \int_0^t\int_\Omega f(u_\tau - u_{xx\tau})\,dx\,d\tau \int_0^t [a(1,\tau)u_x(1,\tau)u_\tau(1,\tau) - a(0,\tau)u_x(0,\tau)u_\tau(0,\tau)]\,d\tau \\ &\quad + \int_0^t [c(1,\tau)u_{x\tau}(1,\tau)u(1,\tau) - c(0,\tau)u_{x\tau}(0,\tau)u(0,\tau)]\,d\tau. \end{aligned} \qquad (27)$$

By taking into account the expressions for $u_{x\tau}(1,\tau)$ and $u_{x\tau}(0,\tau)$ in the boundary conditions (24) and (25) and by using (9), we represent the fifth term on the left-hand side in relation (27) in the form

$$\begin{aligned}
&2\int_0^t [u_{x\tau}(0,\tau)u_\tau(0,\tau) - u_{x\tau}(1,\tau)u_\tau(1,\tau)]\,d\tau \\
&= 2\lambda\int_0^t [\alpha_1 u_x(0,\tau) + \alpha_2 u_x(1,\tau) + \alpha_3 u_\tau(0,\tau) + \alpha_4 u_\tau(1,\tau) + \alpha_5 u(0,\tau) + \alpha_6 u(1,\tau)]u_\tau(0,\tau)\,d\tau \\
&\quad - 2\lambda\int_0^t [\beta_1 u_x(0,\tau) + \beta_2 u_x(1,\tau) + \beta_3 u_\tau(0,\tau) + \beta_4 u_\tau(1,\tau) + \beta_5 u(0,\tau) + \beta_6 u(1,\tau)]u_\tau(1,\tau)\,d\tau \\
&= 2\lambda\int_0^t [\alpha_3 u_\tau^2(0,\tau) + (\alpha_4 - \beta_3)u_\tau(0,\tau)u_\tau(1,\tau) - \beta_4 u_\tau^2(1,\tau)]\,d\tau + J \\
&= 2\lambda\int_0^t q(u_\tau(0,\tau), u_\tau(1,\tau), \tau)\,d\tau + J \ge J, \qquad (28)
\end{aligned}$$

where

$$\begin{aligned}
J = {}& 2\lambda\int_0^t [\alpha_1 u_x(0,\tau) + \alpha_2 u_x(1,\tau) + \alpha_5 u(0,\tau) + \alpha_6 u(1,\tau)]u_\tau(0,\tau)\,d\tau \\
&- 2\lambda\int_0^t [\beta_1 u_x(0,\tau) + \beta_2 u_x(1,\tau) + \beta_5 u(0,\tau) + \beta_6 u(1,\tau)]u_\tau(1,\tau)\,d\tau.
\end{aligned}$$

Note that the nonnegativity of the quadratic form (9) takes place, for example, if either $\alpha_3(t) > 0$ or $\beta_4(t) < 0$ and $\alpha_3(t)\beta_4(t) - (\alpha_4(t) - \beta_3(t))^2 \ge 0$.

By using conditions (9), (10), and (13), the Young inequality, and inequality (23), and by applying the Gronwall lemma, from (28), we finally obtain the a priori estimate

$$\begin{aligned}
&\int_0^t\int_\Omega [u_\tau^2 + 2u_{x\tau}^2 + u_{xx\tau}^2]\,dx\,d\tau + \int_\Omega [u_{xx}^2(x,t) + u_x^2(x,t) + u^2(x,t)]\,dx \\
&\qquad + \int_0^t q(u_\tau(0,\tau), u_\tau(1,\tau), \tau)\,d\tau \le K\int_0^T\int_\Omega f^2\,dx\,d\tau, \qquad (29)
\end{aligned}$$

where K is a constant depending on the number T and the functions $a(x,t)$, $c(x,t)$, $\alpha_i(t)$, and $\beta_j(t)$.

Let us show that the estimate (29) implies that Λ is a closed set.

Let $\{\lambda_n\}$ be a sequence of points of Λ converging to a number λ_0, let $\{w_n(x,t)\}$ be the sequence of solutions of the boundary value problem (20)–(22) with $\lambda = \lambda_n$ in the space V, and let

$$u_n(x,t) = w_n(x,t) + \int_0^t F(x,\tau,\lambda_n,\vec{w}_n(\tau))\,d\tau.$$

Set $v_{nk}(x,t) = u_n(x,t) - u_k(x,t)$. The functions $v_{nk}(x,t)$ satisfy the relations

$$\begin{aligned}
&v_{nkt} - av_{nkxx} + cv_{nk} - v_{nkxxt} = 0, \qquad v_{nkxt}(0,t) = \lambda_k\vec{\alpha}\vec{v}_{nk}(0,t) + (\lambda_n - \lambda_k)\vec{\alpha}\vec{u}_n, \\
&v_{nkx}(1,t) = \lambda_k\vec{\beta}\vec{v}_{nk}(1,t) + (\lambda_n - \lambda_k)\vec{\beta}\vec{u}_n,
\end{aligned}$$

where
$$\vec{u}_n = (u_{nx}(0,t), u_{nx}(1,t), u_{nt}(0,t), u_{nt}(1,t), u_n(0,t), u_n(1,t)).$$

By reproducing the derivation of the estimate (29) for the functions $v_{nk}(x,t)$ and by taking into account the fact that the estimate (29) holds for the functions $u_n(x,t)$ themselves, one can readily show that
$$\|v_{nk}\|_V \le C_1|\lambda_n - \lambda_k|,$$
where C_1 is a constant depending on the functions $a(x,t)$, $c(x,t)$, $\alpha_i(t)$, $\beta_j(t)$, and $f(x,t)$ and the number T. It follows from this inequality that $\{u_n(x,t)\}$ is a Cauchy sequence in the space V. In turn, the Cauchy property implies that there exists a function $u_0(x,t)$ such that $u_0(x,t) \in W_2^{2,1}(Q) \cap L_\infty(0,T;W_2^2(\Omega))$ and $u_{0xxt}(x,t) \in L_2(Q)$; in addition, the functions $u_0(x,t)$ and $u_{0xxt}(x,t)$ are the limits of the sequences $\{u_n(x,t)\}$ and $\{u_{nxxt}(x,t)\}$ in the corresponding spaces. Obviously, the function $u_0(x,t)$ is a solution of the boundary value problem (1), (24)–(26) for $\lambda = \lambda_0$. Set

$$w_0(x,t) = u_0(x,t) - \int_0^t [\gamma_1 u_{0x}(0,t) + \gamma_2 u_{0x}(1,t) + \gamma_3 u_{0t}(0,t) + \gamma_4 u_{0t}(1,t) + \gamma_5 u_0(0,t) + \gamma_6 u_0(1,t)]\, d\tau,$$

where
$$\gamma_k(x,t,\lambda_0) = \frac{\lambda_0 x^2}{2}[\beta_k(t) - \alpha_k(t)] + \lambda_0 x \alpha_k(t), \qquad k = 1, \dots, 6.$$

Obviously, the function $w_0(x,t)$ belongs to the space V and is a solution of the boundary value problem (20)–(22) corresponding to the value $\lambda = \lambda_0$. It follows that λ_0 is a point of the set Λ. The fact that a limit point of this set also belongs to it implies that the set is closed.

Now let us show that Λ is open. Let λ_0 be a point of Λ, and let $\lambda = \lambda_0 + \widetilde{\lambda}$. Let us show that, for small $|\widetilde{\lambda}|$, the number λ belongs to Λ as well.

Set
$$\widetilde{\Phi}(x,t,\lambda,\xi) = \Phi(x,t,\lambda,\xi) - \Phi(x,t,\lambda_0,\xi).$$
Let $v(x,t)$ be a function in the space V. Consider the following boundary value problem: find a function $w(x,t)$ that is a solution of the equation

$$Lw = f + \Phi(x,t,\lambda_0,\vec{w}) + \widetilde{\Phi}(x,t,\lambda,\vec{v}) \tag{30}$$

in the rectangle Q and satisfies conditions (21) and (22). We have the inequality

$$\|\widetilde{\Phi}(x,t,\lambda,\vec{v})\|_{L_2(Q)} \le M|\widetilde{\lambda}|\, \|v\|_V \tag{31}$$

with a constant M depending only on the functions $a(x,t)$, $c(x,t)$, $\alpha_i(t)$, and $\beta_j(t)$. [One can readily prove this inequality by using condition (13), the finite increment theorem, and inequality (23).] Inequality (31), together with the assumption that the number λ_0 belongs to the set Λ, implies that the boundary value problem (30), (21), (22) has a solution $w(x,t)$ in the space V. Consequently, the boundary value problem (30), (21), (22) generates an operator G mapping the space V into itself, $G(v) = w$. By reproducing the derivation of the estimate (29) for this boundary value problem for the case in which $f(x,t) \equiv 0$ and by taking into account inequality (31), we find that if the condition $C_0M|\widetilde{\lambda}| < 1$ is satisfied, then G is a contraction operator. Obviously, the fixed point of the operator G is a solution of the equation

$$Lw = f + \Phi(x,t,\lambda,\vec{w}).$$

Consequently, for small $|\widetilde{\lambda}|$ (more precisely, if the above-represented inequality $C_0M|\widetilde{\lambda}| < 1$ holds), the number λ belongs to the set Λ. It follows that Λ is an open set.

Thus, the set Λ thus defined is nonempty, open, and closed. Consequently, it coincides with the entire interval $[0,1]$. In other words, if conditions (9), (10), and (13) hold, then the boundary value problem (20)–(22) has a solution $w(x,t)$ in the space V for all λ, including $\lambda = 1$.

Set

$$u(x,t) = w(x,t) + \int_0^t F(x,\tau,1,\vec{w}(\tau))\,d\tau.$$

In other words, under conditions (9), (10), and (13), the boundary value problem (1), (11), (12), (3) has a solution $u(x,t)$ that belongs to the space V for all values of λ including $\lambda = 1$.

The uniqueness of solutions of the boundary value problem (1), (11), (12), (3) in the space V follows from the estimate (29).

The proof of the lemma is complete. Note that the above-proved lemma is of interest in itself.

5. PROOF OF THE THEOREM

Let us return to the original problem (1)–(3). We introduce the notation

$$\varphi_p(t) = \frac{1}{2}\int_0^1 x^2 K_p(x,t)\,dx, \qquad \psi_p(t) = \int_0^1 xK_p(x,t)\,dx,$$

$$\varphi_{p+2}(t) = \frac{1}{2}\int_0^1 x^2 N_p(x,t)\,dx, \qquad \psi_{p+2}(t) = \int_0^1 xN_p(x,t)\,dx, \qquad p = 1,2.$$

Just as above, for $(x,t) \in \overline{Q}$, we set

$$\gamma_k(x,t) = \frac{x^2}{2}[\beta_k(t) - \alpha_k(t)] + x\alpha_k(t), \qquad k = 1,\dots,6,$$

and for $\lambda \in [0,1]$ we introduce the functions

$$\begin{aligned} v(x,t) = \lambda \int_0^t \Bigg[&\gamma_1 u_x(0,\tau) + \gamma_2 u_x(1,\tau) + \gamma_3 u_t(0,\tau) + \gamma_4 u_t(1,\tau) + \gamma_5 u(0,\tau) + \gamma_6 u(1,\tau) \\ &+ \left(x - \frac{x^2}{2}\right)\int_0^1 K_1(x,\tau)u(x,\tau)\,dx + \frac{x^2}{2}\int_0^1 K_2(x,\tau)u(x,\tau)\,dx \\ &+ \left(x - \frac{x^2}{2}\right)\int_0^1 N_1(x,\tau)u_t(x,\tau)\,dx + \frac{x^2}{2}\int_0^1 N_2(x,\tau)u_t(x,\tau)\,dx\Bigg]\,d\tau. \end{aligned} \tag{32}$$

Note that the function $v(x,t)$ thus defined obviously satisfies the boundary conditions of the form (6), (7).

Consider the function $w(x,t) = u(x,t) - v(x,t)$, where $u(x,t)$ is the solution of the boundary value problem (1), (6), (7), (3). By substituting $u(x,t) = w(x,t) + v(x,t)$ into the original equation (1), we obtain

$$w_t - a(x,t)w_{xx} + c(x,t)w - w_{xxt} = f(x,t) - v_t + a(x,t)v_{xx} - c(x,t)v + v_{xxt}. \tag{33}$$

We introduce the notation

$$(u_1(t), u_2(t), u_3(t), u_4(t), u_5(t), u_6(t), u_7(t), u_8(t), u_9(t), u_{10}(t))$$
$$= \Bigg(u_x(0,t), u_x(1,t), u_t(0,t), u_t(1,t), u(0,t), u(1,t),$$
$$\int_0^1 K_1(x,t)u(x,t)\,dx, \int_0^1 K_2(x,t)u(x,t)\,dx, \int_0^1 N_1(x,t)u_t(x,t)\,dx, \int_0^1 N_2(x,t)u_t(x,t)\,dx\Bigg),$$
$$(w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10})$$
$$= \Bigg(w_x(0,t), w_x(1,t), w_t(0,t), w_t(1,t), w(0,t), w(1,t),$$
$$\int_0^1 K_1(x,t)w(x,t)\,dx, \int_0^1 K_2(x,t)w(x,t)\,dx, \int_0^1 N_1(x,t)w_t(x,t)\,dx, \int_0^1 N_2(x,t)w_t(x,t)\,dx\Bigg).$$

We determine the function $v(x,t)$ via the newly introduced function $w(x,t)$ from the system of equations

$$w_x(0,t) = u_x(0,t) - \int_0^t \sum_{i=1}^6 \alpha_i(\tau)u_i(\tau)\,d\tau - \int_0^t\int_0^1 [K_1(x,\tau)u(x,\tau) + N_1(x,\tau)u_\tau(x,\tau)]\,dx\,d\tau,$$
$$w_x(1,t) = u_x(1,t) - \int_0^t \sum_{i=1}^6 \beta_i(\tau)u_i(\tau)\,d\tau - \int_0^t\int_0^1 [K_2(x,\tau)u(x,\tau) + N_2(x,\tau)u_\tau(x,\tau)]\,dx\,d\tau,$$
$$w_t(0,t) = u_t(0,t),$$
$$w_t(1,t) = u_t(1,t) - \frac{1}{2}\sum_{i=1}^6 (\alpha_i(t) + \beta_i(t))u_i(t)$$
$$- \frac{1}{2}\int_0^1 [K_1(x,t)u(x,t) + N_1(x,t)u_t(x,t) + K_2(x,t)u(x,t) + N_2(x,t)u_t(x,t)]\,dx,$$
$$w(0,t) = u(0,t),$$
$$w(1,t) = u(1,\tau) - \frac{1}{2}\int_0^t \sum_{i=1}^6 (\alpha_i(\tau) + \beta_i(\tau))u_i(\tau)\,d\tau$$
$$- \frac{1}{2}\int_0^t\int_0^1 [K_1(x,\tau)u(x,\tau) + N_1(x,\tau)u_\tau(x,\tau) + K_2(x,\tau)u(x,\tau) + N_2(x,\tau)u_\tau(x,\tau)]\,dx\,d\tau,$$

$$\int_0^1 K_1(x,t)w(x,t)\,dx = \int_0^1 K_1(x,t)u(x,t)\,dx - \varphi_1(t)\int_0^t \sum_{i=1}^6 [\beta_i(\tau) - \alpha_i(\tau)]u_i(\tau)\,d\tau$$
$$- \psi_1(t)\int_0^t \sum_{i=1}^6 \alpha_i(\tau)u_i(\tau)\,d\tau - [\psi_1(t) - \varphi_1(t)]\int_0^t\int_0^1 [K_1(x,\tau)u(x,\tau) + N_1(x,\tau)u_\tau(x,\tau)]\,dx\,d\tau$$
$$- \varphi_1(t)\int_0^t\int_0^1 [K_2(x,\tau)u(x,\tau) + N_2(x,\tau)u_\tau(x,\tau)]\,dx\,d\tau,$$

$$\int_0^1 K_2(x,t)w(x,t)\,dx = \int_0^1 K_2(x,t)u(x,t)\,dx - \varphi_2(t)\int_0^t \sum_{i=1}^6 [\beta_i(\tau)-\alpha_i(\tau)]u_i(\tau)\,d\tau$$

$$- \psi_2(t)\int_0^t \sum_{i=1}^6 \alpha_i(\tau)u_i(\tau)\,d\tau - [\psi_2(t)-\varphi_2(t)]\int_0^t\int_0^1 [K_1(x,\tau)u(x,\tau)+N_1(x,\tau)u_\tau(x,\tau)]\,dx\,d\tau$$

$$- \varphi_2(t)\int_0^t\int_0^1 [K_2(x,\tau)u(x,\tau)+N_2(x,\tau)u_\tau(x,\tau)]\,dx\,d\tau,$$

$$\int_0^1 N_1(x,t)w_t(x,t)\,dx = \int_0^1 N_1(x,t)u_t(x,t)\,dx - \varphi_3(t)\sum_{i=1}^6 [\beta_i(t)-\alpha_i(t)]u_i(t)$$

$$- \psi_3(t)\sum_{i=1}^6 \alpha_i(t)u_i(t) - [\psi_3(t)-\varphi_3(t)]\int_0^1 [K_1(x,t)u(x,t)+N_1(x,t)u_t(x,t)]\,dx$$

$$- \varphi_3(t)\int_0^1 [K_2(x,t)u(x,t)+N_2(x,t)u_t(x,t)]\,dx,$$

$$\int_0^1 N_2(x,t)w_t(x,t)\,dx = \int_0^1 N_2(x,t)u_t(x,t)\,dx - \varphi_4(t)\sum_{i=1}^6 [\beta_i(t)-\alpha_i(t)]u_i(t)$$

$$- \psi_4(t)\sum_{i=1}^6 \alpha_i(t)u_i(t) - [\psi_4(t)-\varphi_4(t)]\int_0^1 [K_1(x,t)u(x,t)+N_1(x,t)u_t(x,t)]\,dx$$

$$- \varphi_4(t)\int_0^1 [K_2(x,t)u(x,t)+N_2(x,t)u_t(x,t)]\,dx. \tag{34}$$

From the fourth equation in system (34), we find $u_4(t) \equiv u_t(1,t)$. If the condition $p_1(t) \equiv 2-\alpha_4(t)-\beta_4(t) \neq 0$ is satisfied, then from (8), we obtain

$$u_4(t) = \frac{1}{p_1(t)}\Bigg\{2w_t(1,t) + \sum_{\substack{i=1\\ i\neq 4}}^6 (\alpha_i(t)+\beta_i(t))u_i(t)$$

$$+ \int_0^1 [K_1(x,t)u(x,t)+N_1(x,t)u_t(x,t)+K_2(x,t)u(x,t)+N_2(x,t)u_t(x,t)]\,dx\Bigg\}.$$

From the ninth and tenth equations in system (34), we find

$$u_9(t) \equiv \int_0^1 N_1(x,t)u_t(x,t)\,dx, \qquad u_{10}(t) \equiv \int_0^1 N_2(x,t)u_t(x,t)\,dx.$$

Under the condition

$$p_2(t) \equiv 2-\alpha_4(t)-\beta_4(t)-4\varphi_4(t)(1-\alpha_4(t))-\psi_4(t)(2-\beta_4(t)+\alpha_4(t)) \neq 0,$$

from (8), we have

$$\begin{aligned} u_9(t) &= \frac{1}{p_2(t)}[w_9 p_1(t) + 2\varphi_3(t)(\beta_4(t) - \alpha_4(t) + 2\psi_3(t)\alpha_4(t))w_4 \\ &\quad + (2\varphi_3(t)(1-\alpha_4(t))(\vec{\alpha}_0 + \vec{\beta}_0) + \psi_3(t)((2-\beta_4(t))\vec{\alpha}_0 + \alpha_4\vec{\beta}_0))\vec{u}_0], \\ u_{10}(t) &= \frac{1}{p_2(t)}[w_9 p_1(t) + 2\varphi_4(t)(\beta_4(t) - \alpha_4(t) + 2\psi_4(t)\alpha_4(t))w_4 \\ &\quad + (2\varphi_4(t)(1-\alpha_4(t))(\vec{\alpha}_0 + \vec{\beta}_0) + \psi_4(t)((2-\beta_4(t))\vec{\alpha}_0 + \alpha_4\vec{\beta}_0))\vec{u}_0], \end{aligned} \tag{35}$$

where

$$\begin{aligned} \vec{\alpha}_0 &= (\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, 1, 0), \qquad \vec{\beta}_0 = (\beta_1, \beta_2, \beta_3, \beta_5, \beta_6, 0, 1), \\ \vec{u}_0(t) &= (u_1(t), u_2(t), u_3(t), u_5(t), u_6(t), u_7(t), u_8(t)) \\ &= \left(u_x(0,t), u_x(1,t), u_t(0,t), u(0,t), u(1,t), \int_0^1 K_1(x,t)u(x,t)\,dx, \int_0^1 K_2(x,t)u(x,t)\,dx\right). \end{aligned}$$

By eliminating the functions $u_4(t)$, $u_9(t)$, and $u_{10}(t)$ found above from system (34), we obtain the system of equations

$$\vec{w}_0(t) = \vec{u}_0(t) - \int_0^t \vec{G}_0(\tau)\,d\tau, \qquad \vec{G}_0(t) = (G_{01}(t), G_{02}(t), 0, 0, G_{05}(t), G_{06}(t), G_{07}(t)), \tag{36}$$

where

$$\begin{aligned} \vec{w}_0(t) &= (w_1(t), w_2(t), w_3(t), w_5(t), w_6(t), w_7(t), w_8(t)) \\ &= \left(w_x(0,t), w_x(1,t), w_t(0,t), w(0,t), w(1,t), \int_0^1 K_1(x,t)w(x,t)\,dx, \int_0^1 K_2(x,t)w(x,t)\,dx\right), \\ G_{0k}(t) &= \frac{1}{p_1(t)p_2(t)} G_{0k}(w_4(t), w_9(t), w_{10}(t), \vec{u}_0(t)) \qquad (k = 1, 2, 5, 6, 7). \end{aligned}$$

We rewrite system (36) in the form

$$\vec{w}_0(t) = \vec{u}_0(t) - \int_0^t C(\tau)\vec{u}_0(\tau)\,d\tau - \int_0^t [\vec{c}_1 w_4(\tau) + \vec{c}_2 w_9(\tau) + \vec{c}_3 w_{10}(\tau)]\,d\tau, \tag{37}$$

where

$$C(t) \equiv \frac{1}{p_1(t)p_2(t)}(a_{ij})$$

is a seventh-order matrix whose row vectors are defined in closed form just as above and $a_{ij} = a_{ij}(\alpha_4, \beta_4, \varphi_3, \varphi_4, \psi_3, \psi_4, \vec{\alpha}_0, \vec{\beta}_0)$, c_k are seven-dimensional vectors; moreover,

$$\vec{c}_k = \vec{c}_k(\alpha_4, \beta_4, \varphi_3, \varphi_4, \psi_3, \psi_4, \vec{\alpha}_0, \vec{\beta}_0) \qquad (k = 1, 2, 3).$$

The integral equation (37) is a Volterra integral equation of the second kind [11, p. 102; 12, p. 110] and has a unique solution, which can be represented in the form

$$\vec{u}_0(t) = \vec{w}_0(t) + \int_0^t \vec{F}_0(\tau, w_4(\tau), w_9(\tau), w_{10}(\tau), \vec{w}_0(\tau))\,d\tau, \qquad \vec{F}_0 = (F_{01}, F_{02}, 0, 0, F_{05}, F_{06}, F_{07}).$$

By substituting the functions $\vec{u}_0(t)$, $u_4(t)$, $u_9(t)$, and $u_{10}(t)$ found above from (35) into the representation (32), we obtain

$$v(x,t) = \int\limits_0^t F(x,\tau,\lambda,\vec{w}(\tau))\,d\tau.$$

By substituting this expression for $v(x,t)$ into Eq. (33), we finally derive the identity

$$\begin{aligned} w_t - a(x,t)w_{xx} + c(x,t)w - w_{xxt} \\ &= f(x,t) - F(x,t,\lambda,\vec{w}(t)) + a(x,t)\int\limits_0^t F_{xx}(x,\tau,\lambda,\vec{w}(\tau))\,d\tau \\ &- c(x,t)\int\limits_0^t F(x,\tau,\lambda,\vec{w}(\tau))\,d\tau + F_{xx}(x,t,\lambda,\vec{w}(t)) \\ &\equiv f(x,t) + \Phi(x,t,\lambda,\vec{w}(t)). \end{aligned} \tag{38}$$

Consider the following auxiliary boundary value problem: find a function $w(x,t)$ that is a solution of Eq. (38) in the rectangle Q and satisfies the conditions

$$w(x,0) = 0, \qquad x \in \Omega, \tag{39}$$
$$w_{xt}(0,t) = w_{xt}(1,t) = 0, \qquad t \in (0,T). \tag{40}$$

Note that for $\lambda = 1$ the boundary value problem (38)–(40) is equivalent to the original boundary value problem (1), (6), (7), (3).

Next, just as in the proof of the auxiliary boundary value problem 2, under conditions (8)–(10), this boundary value problem has a solution in the space V, which completes the proof.

ACKNOWLEDGMENTS

The author is grateful to A.I. Kozhanov for support and constant attention to the research.

The research was supported by the Federal Targeted Program “Scientific and Pedagogical Staff of Innovational Russia” for 2009–2013 (contract no. 14.132.21.1349).

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