Generalized Multiscale Finite Element Method for Elasticity Problem in Fractured Media

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Abstract. In this work, we consider the elasticity problem in fractured media. For the efficient numerical solution, we present a Generalized Multiscale Finite Element Method (GMsFEM). GMsFEM is used for the construction of a coarse grid approximation of the problem by solution of the local spectral problems. We consider two types of the multiscale basis functions: (1) CG-GMsFEM with continuous multiscale basis functions and (2) DG-GMsFEM with discontinuous multiscale basis functions. The result of the numerical solution for the two-dimensional model problem is presented to show the performance of the presented multiscale method for fractured media. We compute error between the multiscale solution with the fine-scale solutions by choosing different numbers of multiscale basis functions.

Keywords: Fractured media · Elasticity problem · Discontinuous Galerkin · Multiscale method · GMsFEM

1 Introduction

One important class of multiscale problem consists of problems in fractured domains. Due to the variable sizes and geometries of these fractures, solutions to these problems have multiscale features [1–4]. Direct numerical methods in multiscale media are too expensive and some type of model reduction is needed. The main aim of this work is to use Generalized Multiscale Finite Element Method for elasticity equations in fractured media [5–9].

To approximate the model problem with respect to spatial variables, we use the Galerkin discontinuous method for fine grid. In GMsFEM, we solve problem on a coarse grid where each coarse grid consists of fine-grid blocks. The multiscale basis function construction is local and uses both local snapshot solutions and local spectral problems. To couple multiscale basis functions constructed, we consider two methods Continuous Galerkin (CG) approach and Discontinuous

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Galerkin (DG) approach based on symmetric interior penalty method for coarse grid approximation [10–13].

The work is organized as follows. In Sect. 2, we present a model problem and fine grid approximation. In Sect. 3, we consider both the CG-GMsFEM and DG-GMsFEM for coarse grid approximation. Finally, numerical results are presented in Sect. 4.

2 Problem Formulation and Fine Grid Approximation

We consider a mathematical model describing the elastic state of a body with allowance for the presence of defects (fracture). We present the formulation of the problem for the stationary elasticity equation in the computational domain Ω [14]

$$\text{div } \sigma(u) = f, \quad x \in \Omega, \tag{1}$$

where $f = f(x, t)$ is a given source term.

The Eq. (1) is supplemented by the relation between the stress tensor σ and the strain tensor ϵ

$$\epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T), \tag{2}$$

$$\sigma(u) = 2\mu\epsilon(u) + \lambda \text{ div } uE \tag{3}$$

where E is the identity tensor, λ and μ are the Lame parameters.

Substituting the relations (2) and (3) in the Eq. (1), we obtain the following elliptic equation

$$\mu\Delta u + (\lambda + \mu)\text{grad div } u = f. \tag{4}$$

To approximate the problem with respect to spatial variables, we use the discontinuous Galerkin method (IPDG). Let E be an edge (face) between the elements K_1 and K_2, then the average and the jump of the vector u on the edge E is given by the following formula.

$$\{u\} = \frac{u\,|_{K_1} + u\,|_{K_2}}{2}, \quad [u] = u\,|_{K_1} - u\,|_{K_2}.$$

Let $\mathcal{T}_h$ be a triangulation of the computational domain Ω and Γ_h be the set of all interior faces between the elements $\mathcal{T}_h$. Let $\Gamma_c \subset \Gamma_h$ be the subset of all faces, where the displacement field is a continuous function. On the fracture surface, we suppose discontinuous displacements.

The variational formulation of the elasticity equation using the discontinuous Galerkin method (IPDG, Interior Penalty Discontinuous Galerkin) in a fractured medium is defined as follows: find $u \in V_h$ such that

$$\sum_{K\in\mathcal{T}^h}\int_K(\sigma(u),\varepsilon(v))dx - \sum_{E\in\Gamma_c}\int_E\{\tau(u)\}\,[v]\,ds$$
$$-\sum_{E\in\Gamma_c}\int_E\{\tau(v)\}\,[u]\,ds + \gamma_f\sum_{E\in\Gamma_c}\int_E(\lambda+2\mu)\,[u]\,[v]\,ds = \sum_{K\in\mathcal{T}^\langle}\int_K fvdx,$$

where γ_f is the penalty parameter and $\tau(u) = \sigma(u)n$.

3 Coarse Grid Approximation Using GMsFEM

For construction of the coarse grid solver, we use a Generalized Multicale Finite Element Method(GMsFEM). In this section we will consider the CG-GMsFEM and DG-GMsFEM.

3.1 Multiscale Basis Functions for CG-GMsFEM

We generate a coarse grid approximation using construction of the multiscale space. Let $\mathcal{T}_H$ is the coarse grid and ω_i is the local domain, where $i = 1, ..., N_v$ and N_v is the number of coarse grid nodes. A local domain ω_i is obtained by the combining all the coarse cells around one vertex of the coarse grid. We begin by the construction of local snapshot space $V_{snap}^{\omega_i}$. The snapshot space is constructed by solution of the following local problems

$$\begin{aligned} -\nabla \cdot \sigma(u_{k1}) &= 0, \ in \ \omega_i \\ u_{k_1} &= (\delta_j, 0)^T, \ on \ \partial\omega_i \end{aligned} \tag{5}$$

and

$$\begin{aligned} -\nabla \cdot \sigma(u_{k2}) &= 0, \ in \ \omega_i \\ u_{k_2} &= (0, \delta_j)^T, \ on \ \partial\omega_i \end{aligned} \tag{6}$$

where δ_j is the fine-grid delta function by

$$\delta_j(x_l) = \begin{cases} 1, & l = j \\ 0, & l \neq j, \end{cases}$$

We write

$$V_{snap}^{\omega_i} = span\{\psi_j^{\omega_i}, j = 1, 2, ..., M^{\omega_i}\},$$

where M^{ω_i} is the number of basis functions in $V_{snap}^{\omega_i}$.

We will perform a dimension reduction on the above snapshot spaces by use of a spectral problem. First, we will need a partition of unity function χ_i for the local domain ω_i. Partition of unity is the piecewise bi-linear function on the coarse grid that equals to 1 at the coarse vertex x_i, equals to 0 at all other coarse vertices and linear between them. Next, we define the local spectral problem on the snapshot space as

$$A\phi^i = \lambda S\phi^i, \tag{7}$$

where the elements the matrices $A = \{a_{i,j} = a(\phi_i, \phi_j)\}$ and $S = \{s_{i,j} = s(\phi_i, \phi_j)\}$ are defined as follow

$$\begin{aligned} a(u,v) = &\sum_{K \in \mathcal{T}_h^{\omega_i}} \int_K (\sigma(u), \varepsilon(v)) dx - \sum_{E \in \Gamma_c^{\omega_i}} \int_E \{\tau(u)\} [v] \, ds \\ &- \sum_{E \in \Gamma_c^{\omega_i}} \int_E \{\tau(v)\} [u] \, ds + \gamma \sum_{E \in \Gamma_c^{\omega_i}} \int_E (\lambda + 2\mu) [u] [v] \, ds \\ s(u,v) = &\sum_{K \in \mathcal{T}_h^{\omega_i}} \int_K (\lambda + 2\mu) uv dx \end{aligned}$$

where $\gamma = R/h$ is the penalty parameter, $R > 0$. To construct a reduced space $V_{snap}^{\omega_i}$, we select the first L_i eigenvectors $\phi_1, \phi_2, ..., \phi_{L_i}$, corresponding to the first L_i smallest eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_{L_i}$. We define

$$\psi_l^{\omega_i} = \sum_{j=1}^{M^{\omega_i}} (\phi_{lj} \psi_j^{\omega_i}), \quad l = 1, 2, ...L_i, \tag{8}$$

where $\phi_l j$ is the j-th component of ϕ_l. The local offline space is then defined as

$$V_{off}^{\omega_i} = span\{\chi_i \psi_l^{\omega_i}, l = 1, 2..., L_i\}.$$

Next, we define the global continuous Galerkin offline space as

$$V_{off} = span\{V_{off}^{\omega_i}, i = 1, 2..., N_v\}.$$

The coarse-scale system can be calculated by projecting the fine-scale matrices onto coarse grid with global projection matrix assembled from the calculated multiscale basis functions

$$R = (R_1, R_2, ..., R_{N_v})^T, R_i = [\phi_1, \phi_2, ..., \phi_{L_i}], \tag{9}$$

where R_i is the local projection matrix and $\phi_l = \chi_i \psi_l^{\omega_i}$. Using the global projection matrix R, we can define the coarse-scale system. After calculation of the coarse-scale solution u_H, we can recover the fine-scale solution.

$$u_{ms} = R^T u_H$$

3.2 Multiscale Basis Functions for DG-GMsFEM

We construct the local basis functions required for the DG coarse grid coupling. Let K_i is the local domain, where $i = 1, ..., N_c$ and N_c is the number of coarse grid cells. Let V_H be a finite dimensional function space, which consists of functions on each coarse grid cells. For the local basis functions, a snapshot space $V_{snap}^{K_i}$ is first constructed for each coarse grid cells $K_i \in \mathcal{T}_H$. We have two types of local snapshot spaces, $V_{snap}^{K_i} = V_{i,snap}^{K_i} + V_{b,snap}^{K_i}$.

We perform a dimension reduction on the above snapshot spaces by use of a spectral problem. As we mentioned, we construct two types of the multiscale space V_H, $V_H = V_{H,b}^K + V_{H,i}^K$.

Boundary Basis Functions. The first space $V_{H,b}^K$ is constructed in the local snapshot space $V_{b,snap}^{K_i}$. The snapshot space $V_{b,snap}^{K_i}$ is constructed by solution of the following local problems for each-grid edge e_l on the boundary of K_i

$$\begin{aligned} &-\nabla \cdot \sigma(\psi_j^{K_i}) = 0, \ in \ K_i \\ &\psi_j^{K_i} = (\delta_j, 0)^T \ or \ (0, \delta_j)^T, \ on \ \partial K_i \end{aligned} \tag{10}$$

where δ_j is the fine-grid delta function.

We write

$$V_{b,snap}^{K_i} = span\{\psi_j^{K_i}, j = 1, 2, ..., M^{K_i}\},$$

where M^{K_i} is the number of basis functions in $V_{b,snap}^{K_i}$. Then, we define the local spectral problem on the snapshot space

$$A\phi^b = \lambda^b S\phi^b, \tag{11}$$

where $\phi^b \in V_{b,snap}^{K_i}$, $A = \{a_{i,j}\}$ and $S = \{s_{i,j}\}$.

To construct a reduced space $V_{b,H}(K_i)$, we select the first M_b eigenvectors $\phi_1, \phi_2, ..., \phi_{M_b}$, corresponding to the first M_b smallest eigenvalues $\lambda_1^b \leq \lambda_2^b \leq ... \leq \lambda_{M_b}^b$. We define the space $V_{b,H}(K_i)$ by

$$V_{b,H}(K_i) = span\{\phi_{l,K_i}^b, l = 1, 2..., M_b\}$$

These multiscale basis functions from $V_{b,H}(K_i)$ are called boundary basis functions.

Interior Basis Functions. The second space $V_{H,i}^{K_i}$ is defined to capture interior eigenmodes for K_i. The local snapshot space $V_{i,snap}^{K_i}$ for the coarse grid cells K is defined as $V_{i,snap}^{K_i} = V_h^0(K_i)$. We use following spectral problem with homogeneous Dirichlet boundary conditions to identify the important modes

$$\begin{aligned} a(\phi^{K_i,i}, v) &= \lambda^i s(\phi^{K_i,i}, v), \ in \ \in K_i, \\ \phi^{K_i,i} &= (0,0), \ on \ \partial K_i. \end{aligned} \tag{12}$$

To construct a reduced space $V_{i,H}(K_i)$, we select the first M_i eigenvectors $\phi_1, \phi_2, ..., \phi_{M_i}$, corresponding to the first M_i smallest eigenvalues $\lambda_1^i \leq \lambda_2^i \leq ... \leq \lambda_{M_i}^i$. We define the space $V_{i,H}(K_i)$ by

$$V_{i,H}(K_i) = span\{\phi_{l,K_i}^i, l = 1, 2..., M_i\}$$

These multiscale basis functions from $V_{i,H}(K_i)$ are called interior basis functions.

The coarse-scale system can be calculated by projecting the fine-scale matrices onto the coarse grid with the global projection matrix assembled from the calculated multiscale basis functions

$$R = (R_1, R_2, ..., R_N)^T, \quad R_i = \left[\phi_1^{K,i}, \phi_2^{K,i}, ..., \phi_{M_i}^{K,i}, \phi_1^{K,b}, \phi_2^{K,b}, ..., \phi_{M_b}^{K,b}\right], \tag{13}$$

where R_i is the local projection matrix in a coarse grid element K_i and N is the number of coarse grid elements. Using the global projection matrix R, we can define the coarse-scale system After calculation of the coarse-scale solution u_H, we can recover the fine-scale solution.

$$u_{ms} = R^T u_H. \tag{14}$$

4 Numerical Results

We consider numerical simulation using the proposed models. Geometric regions with a computational mesh are shown in Fig. 1. The computational domain $\Omega = [0, L_x] \times [0, L_y]$ c $L_x = L_y = 1$. The computational mesh contains about 13825 vertices and 27248 triangular cells.

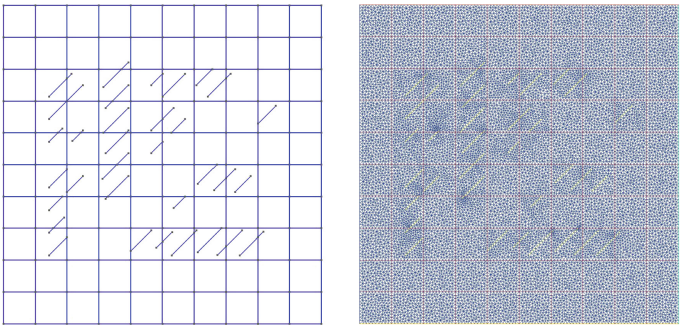

Fig. 1. Computational domain with a computational mesh.

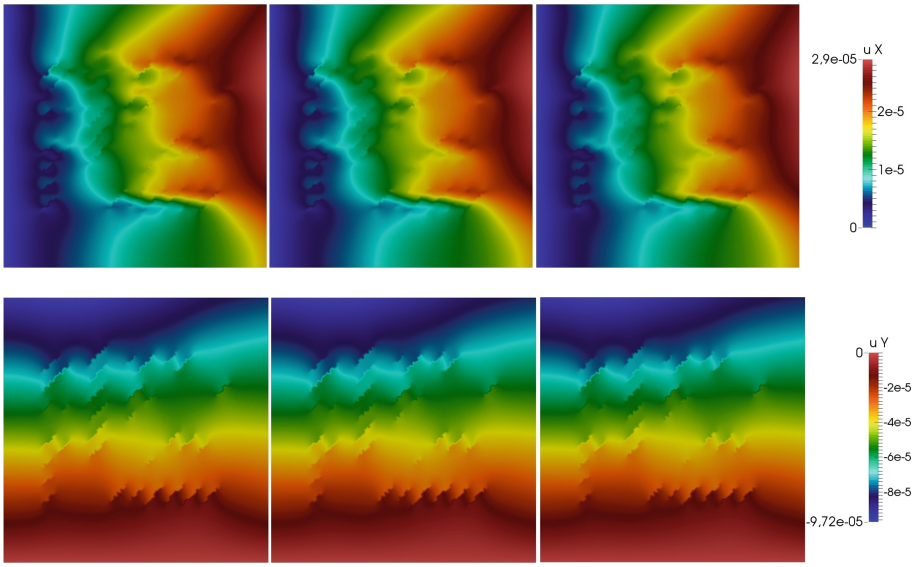

Fig. 2. The distribution of the X and Y components of the displacements (u_x - first row and u_y - second row). First column: fine grid solution. Second column: multiscale solution using CG-GMsFEM. Third column: multiscale solution using DG-GMsFEM

Table 1. Relative errors for solutions with different number of multiscale basis functions using CG-GMsFEM. Left: 5×5 coarse grid. Right: 10×10 coarse grid

Basis	L_2 norm (%)	H_1 norm (%)
1	95.567	93.632
2	82.387	79.005
4	66.210	62.535
6	55.996	55.301
8	6.017	17.178
12	2.767	8.487
16	2.497	4.629

Basis	L_2 norm (%)	H_1 norm (%)
1	95.747	93.455
2	19.896	30.377
4	11.938	21.760
6	6.028	13.158
8	1.591	8.597
12	0.400	4.447
16	0.170	2.625

Table 2. Relative errors for displacement for different number of multiscale basis functions for DG-GMsFEM

$M_i = 0, M_b$	L_2 norm (%)	H_1 norm (%)
5	44.614	51.015
10	7.946	17.249
15	4.183	12.409
20	2.195	8.275
25	1.662	7.099

$M_i + M_b$	L_2 norm (%)	H_1 norm (%)
5	42.525	48.648
10	5.422	12.915
15	2.395	8.685
20	1.440	6.673
25	1.092	5.825

For the calculations, we used the following parameters: $\mu = 1$, $\lambda = 1$ and source term $f = (0, 0)$. We set the following boundary conditions

$$\begin{aligned} u_x &= 0, x \in \Gamma_1 \\ u_y &= 0, x \in \Gamma_4 \\ u \cdot n &= g, x \in \Gamma_2 \end{aligned} \tag{15}$$

where $g = (0, -0.0002)$. Here Γ_1, Γ_4 are the upper and lower boundaries, Γ_2 and Γ_3 are the right and left boundaries.

The results of calculations in the region with fractures using the mathematical model of (4), (15) are shown in Fig. 2. In Fig. 2, we present the X and Y components of displacement distributions for the numerical solution using CG-GMsFEM, DG-GMsFEM and using fine grid approximation. Calculations was performed using 25 multiscale basis functions for DG-GMsFEM and 16 multiscale functions for CG-GMsFEM.

Relative L_2 errors for CG-GMsFEM are shown in Table 1 for 5×5 and 10×10 coarse grids. Relative L_2 errors for DG-GMsFEM are shown in Table 2, where M_i and M_b are the numbers of the multiscale interior and boundary basis functions. From the numerical results, we observe that the interior multiscale basis functions can reduce errors. We see that a solution using CG-GMsFEM bases is better than solution using DG-GMsFEM.

We present numerical results for fractured domain to demonstrate a robustness of proposed method with two types of the multiscale basis functions. We construct reduced order model using Generalized Multiscale Finite Element Method

using continuous and discontinuous multiscale basis functions. Our results show that the presented method demonstrate high accuracy for elasticity problem.

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