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Smooth Solutions of Parabolic Equations with Changing Time Direction

Ivan E. Egorov[1,b)] and Sergey V. Popov[1,a)]

[1]*Ammosov North-Eastern Federal University, 58, Belinsky str., Yakutsk, 677000, Russia*

[a)]Corresponding author: guspopov@mail.ru
[b)]ivanegorov51@mail.ru

Abstract. We prove solvability of boundary value problems for forward-backward parabolic equations with a full matrix of gluing conditions. As is known, in the case of forward-backward equations the smoothness of initial and boundary data does not ensure Hölder smoothness of solutions. It is shown that the Hölder classes of solutions to boundary value problem for forward-backward parabolic equations and the number of solvability conditions depend on the matrix of gluing (conjugation) conditions.

INTRODUCTION

For nonclassical boundary value problems, smoothness of initial and boundary data does not ensure the Hölder smoothness of solutions. Applying the theory of singular integral equations allows us to indicate smoothness of the data of the problem and necessary and sufficient conditions for Hölder smoothness of solutions. A unified approach with general gluing conditions for these equations allows us to demonstrate that the nonintegral exponent of a Hölder space can essentially influence the number of solvability conditions and smoothness of solutions to a nonclassical equations.

In some simplest cases S.A. Tersenov describe necessary and sufficient solvability conditions of these problems for second order parabolic equations in the spaces $H_{x\,t}^{p,p/2}$ with $p > 2$ in [1]. Solvability (orthogonality) conditions for the data of the problems in this article are written out in an explicit form. Note that the number of orthogonality conditions is finite in the one-dimensional case in contrast to the multi-dimensional case [2].

In the present article we consider well-posedness questions for forward-backward $2n$-parabolic equations with $n = 2, 3$ and a full matrix of gluing conditions. It is demonstrated that solutions to boundary value problems depend on a fractional exponent of a Hölder space and entries of the matrix of gluing conditions provided that necessary and sufficient conditions for the input data of the problem hold.

STATEMENT OF THE PROBLEM

In the domain $Q^+ = \mathbb{R}^+ \times (0, T)$ we consider the system of equations

$$u_t^1 = Lu^1, \quad -u_t^2 = Lu^2 \quad \left(L \equiv (-1)^{n+1} \frac{\partial^{2n}}{\partial x^{2n}}\right). \tag{1}$$

A solution to the system (1) is sought in the Hölder space $H_{x\,t}^{p,p/2n}(Q^+)$, $p = 2nl + \gamma$, $0 < \gamma < 1$. It satisfies the initial-boundary conditions

$$u^1(x, 0) = \varphi_1(x), \quad u^2(x, T) = \varphi_2(x), \ x > 0, \tag{2}$$

and the gluing conditions

$$\vec{u}^{\,1}(0, t) = A\,\vec{u}^{\,2}(0, t), \tag{3}$$

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where $\vec{u}^{\,k} = (u^k, u^k_x, \ldots, u^k_{\underbrace{x\ldots x}_{2n-1}})$ and A is a nondegenerate matrix with constant real-valued coefficients.

We assume that $\varphi_1(x), \varphi_2(x) \in H^p(\mathbb{R})$,

$$\omega_1(x,t) = \frac{1}{\pi}\int\limits_{\mathbb{R}} U(x,t;\xi,0)\varphi_1(\xi)\,d\xi, \quad \omega_2(x,t) = \frac{1}{\pi}\int\limits_{\mathbb{R}} U(\xi,T;x,t)\varphi_2(\xi)\,d\xi$$

and employ the following integral representation for a solution to the system (1):

$$u^1(x,t) = \int\limits_0^t \overrightarrow{U_1}(x,t;0,\tau)\vec{\alpha}(\tau)\,d\tau + \omega_1(x,t), \quad u^2(x,t) = \int\limits_t^T \overrightarrow{U_2}(0,\tau;x,t)\vec{\beta}(\tau)\,d\tau + \omega_2(x,t),$$

where $\overrightarrow{U_1}, \overrightarrow{U_2}$ are row vectors, with $\overrightarrow{U_1} = (U, V_1, \ldots, V_{n-1})$, $\overrightarrow{U_2} = (U, W_1, \ldots, W_{n-1})$, U is a fundamental solution, V_p and W_p are elementary Cattabriga solutions [2] of the first equation (1), and $\vec{\alpha}(t)$, $\vec{\beta}(t)$ are column vectors of unknown densities with components $\alpha_p(t), \beta_p(t),\ p = 0, 1, \ldots, n-1$. The functions $\omega_1(x,t)$, $\omega_2(x,t)$ are solutions to the equations (1) satisfying the conditions (2) in $\mathbb{R}$.

Consider the case of the upper triangular matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1,2n-1} & a_{1,2n} \\ 0 & a_{22} & \ldots & a_{2,2n-1} & a_{2,2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_{2n-1,2n-1} & a_{2n-1,2n} \\ 0 & 0 & \ldots & 0 & a_{2n,2n} \end{pmatrix}. \tag{4}$$

We are in the conditions of the article [3] in the case of a symmetric matrix A.

Parabolic Equations of the Forth Order

We consider the equation (1) for $n = 2$. Note that the case of a symmetric matrix A is considered also in [4,5]. We assume that the entries a_{ij} of A satisfy the condition of uniqueness of a solution to the boundary value problem (1)–(3).

Theorem 1. Let the entries of a nondegenerate matrix A of the form (4) satisfy the conditions $a_{ij} = 0$, $i < j \le 5$, $a_{33} - \sqrt{2}a_{22} \neq 0$, and $\varphi_1(x), \varphi_2(x) \in H^p(\mathbb{R}^+)$ $(p = 4l + \gamma)$. Then under the $4l$ conditions

$$L_s(\varphi_1, \varphi_2) = 0, \quad s = 1, \ldots, 4l \tag{5}$$

there exists a unique solution to the equation (1), satisfying the conditions (2), (3) from the space

1) $H^{p,p/4}_{x\ t}(Q^+)$ if $0 < \gamma < 1 - 4\theta$;

2) $H^{q,q/4}_{x\ t}(Q^+)$, $q = 4l + 1 - 4\theta$, if $1 - 4\theta < \gamma < 1$;

3) $H^{q-\varepsilon,(q-\varepsilon)/4}_{x\ \ t}(Q^+)$ if $\gamma = 1 - 4\theta$, where ε is an arbitrary small positive constant, $\theta = \frac{1}{\pi}\arctan|\frac{a}{b}| \in (0, \frac{1}{4})$, $a = a_{33}$, $b = a_{33} - \sqrt{2}a_{22}$.

Theorem 2. Let the entries of a nondegenerate matrix A of the form (4) satisfy the conditions $a_{24} \neq 0$, $\quad a_{34} \neq 0$, and

$$\begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix} \neq 0, \quad \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \neq 0. \tag{6}$$

Assume also that $\varphi_1(x), \varphi_2(x) \in H^p(\mathbb{R}^+)$ $(p = 4l + \gamma)$. Then under $6l + 2$ conditions of the form (5) there exists a unique solution to the equation (1) satisfying the conditions (2), (3) from the space $H^{p,p/4}_{x\ t}(Q^+)$.

Proof of the theorems

The scheme of the proof of Theorem 1. In view of the general results the densities $\alpha_k(t), \beta_k(t)$ $(k = 0, 1)$ must belong to the space $H^q(0,T)$ $(q = \frac{p-3}{4})$ and, moreover,

$$\alpha_k^{(s)}(0) = \beta_k^{(s)}(T) = 0 \quad (s = 0, \ldots, l-1). \tag{7}$$

The gluing conditions (3) generate the following system of integral equations with the Abel operators relative to α_k, β_k:

$$\begin{cases} \frac{1}{4}\Gamma(\frac{1}{4})\int\limits_0^t \frac{\alpha_0(\tau)+\alpha_1(\tau)}{(t-\tau)^{\frac{1}{4}}}\,d\tau + \omega_1(0,t) = a_{11}\left(\frac{1}{4}\Gamma(\frac{1}{4})\int\limits_t^T \frac{\beta_0(\tau)+\beta_1(\tau)}{(\tau-t)^{\frac{1}{4}}}\,d\tau + \omega_2(0,t)\right)+ \\ +a_{12}\left(-\frac{1}{2}\Gamma(\frac{1}{2})\int\limits_t^T \frac{\beta_1(\tau)}{(\tau-t)^{\frac{1}{2}}}\,d\tau + \omega_{2x}(0,t)\right) + a_{13}\left(-\frac{1}{4}\Gamma(\frac{3}{4})\int\limits_t^T \frac{\beta_0(\tau)-\beta_1(\tau)}{(\tau-t)^{\frac{3}{4}}}\,d\tau+ \right. \\ \left. +\omega_{2xx}(0,t)\right) + a_{14}(\frac{\pi}{2}\beta_0(t) + \omega_{2xxx}(0,t)), \\ -\frac{1}{2}\Gamma(\frac{1}{2})\int\limits_0^t \frac{\alpha_1(\tau)}{(t-\tau)^{\frac{1}{2}}}\,d\tau + \omega_{1x}(0,t) + a_{22}\left(-\frac{1}{2}\Gamma(\frac{1}{2})\int\limits_t^T \frac{\beta_1(\tau)}{(\tau-t)^{\frac{1}{2}}}\,d\tau + \omega_{2x}(0,t)\right)+ \\ +a_{23}\left(-\frac{1}{4}\Gamma(\frac{3}{4})\int\limits_t^T \frac{\beta_0(\tau)-\beta_1(\tau)}{(\tau-t)^{\frac{3}{4}}}\,d\tau + \omega_{2xx}(0,t)\right) + a_{24}(\frac{\pi}{2}\beta_0(t) + \omega_{2xxx}(0,t)) = 0, \\ -\frac{1}{4}\Gamma(\frac{3}{4})\int\limits_0^t \frac{\alpha_0(\tau)-\alpha_1(\tau)}{(t-\tau)^{\frac{3}{4}}}\,d\tau + \omega_{1xx}(0,t) = a_{33}\left(-\frac{1}{4}\Gamma(\frac{3}{4})\int\limits_t^T \frac{\beta_0(\tau)-\beta_1(\tau)}{(\tau-t)^{\frac{3}{4}}}\,d\tau+ \right. \\ \left. +\omega_{2xx}(0,t)\right) + a_{34}(\frac{\pi}{2}\beta_0(t) + \omega_{2xxx}(0,t)), \\ \frac{\pi}{2}\alpha_0(t) + \omega_{1xxx}(0,t) + a_{44}(\frac{\pi}{2}\beta_0(t) + \omega_{2xxx}(0,t)) = 0. \end{cases} \tag{8}$$

Since $a_{12} = a_{13} = a_{23} = 0$, excluding, respectively, $\frac{\pi}{2}\beta_0(t) + w_{2xxx}(0,t)$ from the first two equations of the system (8), from the second and third equations, and also from the third and forth equations, we obtain the following system of integral equations of the form (8):

$$\begin{cases} \frac{a_{14}}{2}\Gamma(\frac{1}{2})\int\limits_0^t \frac{\alpha_1(\tau)}{(t-\tau)^{\frac{1}{2}}}\,d\tau + \frac{a_{14}a_{22}}{2}\Gamma(\frac{1}{2})\int\limits_t^T \frac{\beta_1(\tau)}{(\tau-t)^{\frac{1}{2}}}\,d\tau = \\ = \frac{a_{24}}{4}\Gamma(\frac{1}{4})\int\limits_0^t \frac{\alpha_0(\tau)+\alpha_1(\tau)}{(t-\tau)^{\frac{1}{4}}}\,d\tau - \frac{a_{11}a_{24}}{4}\Gamma(\frac{1}{4})\int\limits_t^T \frac{\beta_0(\tau)+\beta_1(\tau)}{(\tau-t)^{\frac{1}{4}}}\,d\tau+ \\ +a_{24}\omega_1(0,t) - a_{11}a_{24}\omega_2(0,t) + a_{14}\omega_{1x}(0,t) + a_{14}a_{22}\omega_{2x}(0,t), \\ \frac{a_{24}}{4}\Gamma(\frac{3}{4})\int\limits_0^t \frac{\alpha_0(\tau)-\alpha_1(\tau)}{(t-\tau)^{\frac{3}{4}}}\,d\tau - \frac{a_{24}a_{33}}{4}\Gamma(\frac{3}{4})\int\limits_t^T \frac{\beta_0(\tau)-\beta_1(\tau)}{(\tau-t)^{\frac{3}{4}}}\,d\tau = \\ = -\frac{a_{34}}{2}\Gamma(\frac{1}{2})\int\limits_0^t \frac{\alpha_1(\tau)}{(t-\tau)^{\frac{1}{2}}}\,d\tau - \frac{a_{22}a_{34}}{2}\Gamma(\frac{1}{2})\int\limits_t^T \frac{\beta_1(\tau)}{(\tau-t)^{\frac{1}{2}}}\,d\tau+ \\ +a_{24}\omega_{1xx}(0,t) - a_{33}a_{24}\omega_{2xx}(0,t) + a_{34}\omega_{1x}(0,t) + a_{34}a_{22}\omega_{2x}(0,t), \\ \frac{\pi}{2}a_{34}\alpha_0(t) = \frac{a_{44}}{4}\Gamma(\frac{3}{4})\int\limits_0^t \frac{\alpha_0(\tau)-\alpha_1(\tau)}{(t-\tau)^{\frac{3}{4}}}\,d\tau - \frac{a_{33}a_{44}}{4}\Gamma(\frac{3}{4})\int\limits_t^T \frac{\beta_0(\tau)-\beta_1(\tau)}{(\tau-t)^{\frac{3}{4}}}\,d\tau- \\ -a_{34}\omega_{1xxx}(0,t) + a_{33}a_{44}\omega_{2xx}(0,t) - a_{44}\omega_{1xx}(0,t), \\ \frac{\pi}{2}(\alpha_0(t) + a_{44}\beta_0(t)) + \omega_{1xxx}(0,t) + a_{44}\omega_{2xxx}(0,t) = 0. \end{cases} \tag{9}$$

For convenience, we assume that $T = 1$. Applying the inversion formula for the Abel operator to the first two equations in (8), we obtain an equivalent system of singular integral equations of the forth order of the form

$$\begin{cases} a_{14}\alpha_1(t) + \frac{a_{14}a_{22}}{\pi}\int\limits_0^1 \left(\frac{\tau}{t}\right)^{\frac{1}{2}}\frac{\beta_1(\tau)}{\tau-t}\,d\tau = a_{24}\frac{\Gamma(\frac{3}{4})}{2\pi}\int\limits_0^t \frac{\alpha_0(\tau)+\alpha_1(\tau)}{(t-\tau)^{3/4}}\,d\tau - \\ -a_{11}a_{24}\frac{\Gamma(\frac{1}{4})}{2\sqrt{\pi^3}}\int\limits_0^1 K_1(t,\tau)(\beta_0(\tau)+\beta_1(\tau))\,d\tau + \frac{d}{dt}\int\limits_0^t \frac{\Phi_0(\tau)}{(t-\tau)^{1/2}}\,d\tau, \\ \sqrt{2}a_{24}(\alpha_0(t)-\alpha_1(t)) - a_{24}a_{33}(\beta_0(t)-\beta_1(t)) - \frac{a_{24}a_{33}}{\pi}\int\limits_0^1 (\frac{\tau}{t})^{\frac{1}{4}}\frac{\beta_0(\tau)-\beta_1(\tau)}{\tau-t}\,d\tau = \\ = -2\frac{a_{34}}{\Gamma(\frac{1}{4})}\int\limits_0^t \frac{\alpha_1(\tau)}{(t-\tau)^{\frac{3}{4}}}\,d\tau - 2\frac{a_{22}a_{34}}{\sqrt{\pi}\Gamma(\frac{3}{4})}\int\limits_0^1 K_2(t,\tau)\beta_1(\tau)\,d\tau + \frac{d}{dt}\int\limits_0^t \frac{\Phi_1(\tau)}{(t-\tau)^{1/4}}\,d\tau, \\ a_{34}\alpha_0(t) = a_{44}\frac{\Gamma(\frac{3}{4})}{2}\int\limits_0^t \frac{\alpha_0(\tau)-\alpha_1(\tau)}{(t-\tau)^{\frac{3}{4}}}\,d\tau - \frac{a_{33}a_{44}}{2\pi}\Gamma(\frac{3}{4})\int\limits_t^1 \frac{\beta_0(\tau)-\beta_1(\tau)}{(\tau-t)^{\frac{3}{4}}}\,d\tau - \Phi_2(t), \\ \alpha_0(t) + a_{44}\beta_0(t) = \Phi_3(t). \end{cases} \tag{10}$$

$$K_1(t,\tau) = \begin{cases} -\frac{2}{3}\left(\frac{\tau}{t}\right)^{\frac{3}{4}}\frac{F\left(\frac{1}{4},\frac{3}{4},\frac{7}{4};\frac{\tau}{t}\right)}{(t-\tau)^{3/4}} = \frac{K_1^0(t,\tau)}{(t-\tau)^{3/4}}, & \tau < t, \\ \left(\frac{\tau}{t}\right)^{\frac{1}{2}}\frac{F\left(-\frac{1}{2},\frac{1}{4},\frac{1}{2};\frac{t}{\tau}\right)}{(\tau-t)^{3/4}} = \frac{K_1^1(t,\tau)}{(\tau-t)^{3/4}}, & \tau > t, \end{cases}$$

$$K_2(t,\tau) = \begin{cases} -\frac{1}{2}\left(\frac{\tau}{t}\right)^{\frac{1}{2}}\frac{F\left(\frac{1}{4},\frac{1}{2},\frac{3}{2};\frac{\tau}{t}\right)}{(t-\tau)^{3/4}} = \frac{K_2^0(t,\tau)}{(t-\tau)^{3/4}}, & \tau < t, \\ \left(\frac{\tau}{t}\right)^{\frac{1}{4}}\frac{F\left(-\frac{1}{4},\frac{1}{4},\frac{3}{4};\frac{t}{\tau}\right)}{(\tau-t)^{3/4}} = \frac{K_2^1(t,\tau)}{(\tau-t)^{3/4}}, & \tau > t. \end{cases}$$

To prove existence of a solution $\alpha_k(t), \beta_k(t)$ $(k = 0, 1)$ to the system (10) lying in the space H^q $(q = (p-3)/4$, $p = 4l+\gamma, 0 < \gamma < 1)$ and satisfying the conditions (7), we reduce the boundary value problem (1)–(3) to the following system of singular integral equations of the normal type:

$$K\vec{\beta} \equiv P\vec{\beta}(t) - \frac{1}{\pi}\int\limits_0^1 \frac{N(t,\tau)\vec{\beta}(\tau)}{\tau-t}\,d\tau = \vec{Q}(t), \tag{11}$$

where

$$\vec{\beta}(t) = (\tilde{\beta}_0^{(l-1)}(t), \tilde{\beta}_1^{(l-1)}(t)),$$

$$\tilde{\beta}_i^{(l-1)}(t) = \beta_i^{(l-1)}(t) - \beta_i^{(l-1)}(0)(1-t),$$

$$P = \begin{pmatrix} a_{14}a_{24}a_{33}a_{34} & -a_{14}a_{24}a_{33}a_{34} \\ -a_{34}a_{44} & 0 \end{pmatrix},$$

$$N(t,\tau) = \begin{pmatrix} a_{14}a_{24}a_{33}a_{34}(\frac{t}{\tau})^{3/4} & a_{14}a_{24}a_{33}a_{34}(\frac{t}{\tau})^{3/4} - \sqrt{2}a_{14}a_{22}a_{24}a_{34}(\frac{t}{\tau})^{1/2} \\ 0 & 0 \end{pmatrix}.$$

The system of singular equations (11) can be rewritten as follows:

$$\begin{cases} a_{33}\tilde{\beta}_1^{(l-1)}(t) - \frac{a_{33}-\sqrt{2}a_{22}}{\pi}\int\limits_0^1 \frac{\tilde{\beta}_1^{(l-1)}(\tau)}{\tau-t}\,d\tau + \frac{1}{\pi}\int\limits_0^1 M(t,\tau)\tilde{\beta}_1^{(l-1)}(\tau)\,d\tau = \bar{q}_1(t), \\ a_{34}a_{44}\tilde{\beta}_0^{(l-1)}(t) = q_2(t). \end{cases} \tag{12}$$

Separating out the characteristic part of the singular equation in (12), we infer

$$a\tilde{\beta}_1^{(l-1)}(t) - \frac{b}{\pi}\int\limits_0^1 \frac{\tilde{\beta}_1^{(l-1)}(\tau)}{\tau - t}\,d\tau = g(t), \quad a = a_{33}, \quad b = a_{33} - \sqrt{2}a_{22}. \tag{13}$$

A solution to the singular integral equation (13) in the class of functions bounded at the ends of the segment (0, 1) is sought with the use of the piecewise analytic function [6,7]

$$\Psi(z) = \frac{1}{2\pi i}\int\limits_0^1 \frac{\tilde{\beta}_1^{(l-1)}(\tau)}{\tau - z}\,d\tau = \frac{\chi(z)}{2\pi i}\int\limits_0^1 \frac{g(\tau)\,d\tau}{(a - bi)\chi^+(\tau)(\tau - z)}\,, \tag{14}$$

where the canonical function $\chi(z) = z^{\theta}(z-1)^{1-\theta}$ if a, b are of different signs and $\chi(z) = z^{1-\theta}(z-1)^{\theta}$ otherwise and $\theta = \frac{1}{\pi}\arctan|\frac{a}{b}|$. Since the index $\varkappa$ of the Riemann-Hilbert problem is equal to -1, the equality (14) is fulfilled under the condition

$$\int_0^1 \frac{g(\tau)}{\chi(\tau)}\,d\tau = 0. \tag{15}$$

In this case, we have

$$\tilde{\beta}_1^{(l-1)}(t) = \Psi^+(t) - \Psi^-(t) = \frac{1}{2}g(t) - \frac{\chi(t)}{2\pi}\int\limits_0^1 \frac{g(\tau)\,d\tau}{\chi(\tau)(\tau - t)}. \tag{16}$$

The formulas (15) can be considered as necessary and sufficient conditions for boundedness of $\tilde{\beta}_1^{(l-1)}(t)$ at $t = 1$.

Inserting the values of $g(t)$ in (16) and taking the first equation in (12) into account, we arrive at the system of Fredholm equations

$$\vec{\beta} + k\vec{\beta} = \vec{Q}^*, \tag{17}$$

where

$$k\vec{\beta} = -\frac{1}{\pi}\int\limits_0^1 N(t,\tau)\vec{\beta}(\tau)\,d\tau.$$

All bounded integrable solutions to the Fredholm system (17) obviously belong to the Hölder space at all points of the contour (0, 1) different form the ends. The properties of the kernel $N(t, \tau)$ and the free term $\vec{Q}^*$ imply that all bounded integrable solutions to the Fredholm systems (17) at the points 0,1 behave as $t^{\frac{1}{2}+\theta}(1-t)^{\frac{1}{2}-\theta}$ provided that a and b are of different signs and as $t^{\frac{1}{2}-\theta}(1-t)^{\frac{1}{2}+\theta}$ otherwise.

By the Mushelishvili theorem about the membership of the Cauchy type integral in the Hölder class at the ends of integration contour (see [8,9]) provided that $\frac{1+\gamma}{4} < \frac{1}{2} - \theta$ $(\theta < \frac{1}{4})$, we establish that solutions to the Fredholm equations (17) belong to the space $H^{\frac{1+\gamma}{4}}(0, 1)$ and vanish at 0, 1 with the order $\frac{1+\gamma}{4}$. Moreover, solutions to the Fredholm equations (17) satisfy the Hölder condition with exponent $\frac{1}{2} - \theta$ for $1 - 4\theta < \gamma < 1$ and the Hölder condition with exponent $\frac{1}{2} - \theta - \varepsilon$ for $\gamma = 1 - 4\theta$.

Thus, the system of equations (17) is equivalent to the initial system of equations (8) provided that the conditions of the form (5) hold.

Solvability of the Fredholm system of equations (17) results from uniqueness of solutions to the main problem (1)–(3) and uniqueness of their representations through potentials. Insert the values of the functions

$$\begin{gathered}\vec{\beta}^{(s)}(t) = \sum_{k=s}^{l-2} \tfrac{\vec{\beta}^{(k)}(0)}{(k-s)!}\, t^{k-s} + \tfrac{1}{(l-2-s)!}\int\limits_0^t (t-\tau)^{l-2-s}\vec{\beta}^{(l-1)}(\tau)\,d\tau, \\ (s = 0, \ldots, l-2)\end{gathered} \tag{18}$$

found by the Taylor formula in our conditions of the form (5). We obtain the $4l$ solvability conditions (5) for the problem (1)–(3) in the space $H^{p,p/4}_{x\ t}(Q^+)$. The proof is complete.

Proof of Theorem 2. In this case the system (11) of singular equations can be rewritten as follows:

$$\begin{cases} \begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix} \cdot \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \cdot \left(\tilde{\beta}_1^{(l-1)}(t) + \frac{1}{\pi} \int\limits_0^1 \frac{\tilde{\beta}_1^{(l-1)}(\tau)}{\tau - t} \, d\tau \right) = \tilde{q}_1(t), \\ \\ a_{34} a_{44} \tilde{\beta}_0^{(l-1)}(t) = \tilde{q}_2(t), \end{cases} \tag{19}$$

where $\theta = \frac{1}{4}$ and $\chi(z) = z^{1/4}(z-1)^{3/4}$ is the canonical function. We are in the conditions of the article [4], namely, if the conditions of Theorem 2 for $\theta \in [\frac{1}{4}, \frac{1}{2})$ hold then a unique solution to the problem (1)—(3) from the space $H_{x\ t}^{p,p/4}(Q^+)$ exists provided that $6l + 2$ conditions of the form (5) hold. The theorem is proven.

Remark.

1. Theorems 1 and 2 remains valid if at least one of the entries a_{13}, a_{14}, and a_{24} of the nondegenerate matrix (4) of gluing conditions is different from zero and the conditions (6) holds. If the nondegenerate matrix (4) of gluing conditions is such that $a_{13} = a_{14} = a_{24} = 0$ then under the conditions (6) the unconditional solvability theorem holds.

2. Similar theorems are valid for $n = 3$ in the case of forward-backward parabolic equations of the six order.

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