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Parabolic Equations with Changing Time Direction and a Full Matrix of Gluing Conditions

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Abstract. Solvability of a boundary value problems for $2n$-parabolic equations with changing time direction in the case of a full matrix of gluing conditions is established. It is shown that Hölder classes of solutions depend both on a noninteger Hölder exponent and the entries of the matrix of gluing conditions under finitely many necessary and sufficient conditions on the data of the problem.

INTRODUCTION

We consider $2n$-parabolic equations with changing time direction and general gluing conditions. For such problems, the smoothness of initial and boundary data does not ensure the membership of a solution in some Hölder space [1]. Application of the theory of singular equations along with the smoothness of the data of the problem makes it possible to find additional necessary and sufficient conditions ensuring that a solution belongs to the Hölder spaces $H_{x\ t}^{p,p/2n}$ for $p \geq 2n$. A unified approach applied under general gluing conditions to such equations shows that the noninteger exponent $p-[p]$ of the Hölder spaces $H_{x\ t}^{p,p/2n}$ can influences essentially both the number of solvability conditions and the smoothness of a solution to a $2n$-parabolic equations with changing time direction.

Conditions ensuring the solvability of boundary value problems for $2n$-parabolic equations with changing time direction are written out explicitly in the articles [2, 3].

To prove $[p]$-solvability for $n = 2$ and $n = 3$, the general diagonal gluing conditions are considered. Moreover, the dependence of the exponents of the Hölder spaces on the weight gluing functions is found and, for $n \geq 4$, it turns out that it suffices to consider continuous gluing conditions including the $(2n-1)$-th derivative on the boundary.

In this article we consider the questions of well-posedness of boundary value problems for $2n$-parabolic equations with changing time direction with the full matrix of gluing conditions. As is shown in [4], the Hölder classes of solutions depend both on a nonintegral Hölder exponent and the entries of a matrix of gluing conditions under necessary and sufficient conditions on the data of the problem.

$2n$-PARABOLIC EQUATIONS

In the domain $Q^+ = \mathbb{R}^+ \times (0,T)$ we consider the system of equations

$$u_t^1 = Lu^1, \quad -u_t^2 = Lu^2 \quad \left(L \equiv (-1)^{n+1}\frac{\partial^{2n}}{\partial x^{2n}}\right). \tag{1}$$

A solution to the equations (1) is sought in the Hölder space $H_{x\ t}^{p,p/2n}(Q^+)$, $p = 2nl + \gamma$, $0 < \gamma < 1$; it satisfies the initial conditions

$$u^1(x,0) = \varphi_1(x), \quad u^2(x,T) = \varphi_2(x), \ x > 0 \tag{2}$$

and the gluing conditions

$$\vec{u}^{\,1}(0,t) = A\,\vec{u}^{\,2}(0,t), \tag{3}$$

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where $\vec{u}^{\,k} = (u^k, u^k_x, \ldots, u^k_{\underbrace{x\ldots x}_{2n-1}})$, and A is a nonsingular matrix with nonzero constant real entries.

Assume that $\varphi_1(x), \varphi_2(x) \in H^p(\mathbb{R})$,

$$\omega_1(x,t) = \frac{1}{\pi}\int\limits_{\mathbb{R}} U(x,t;\xi,0)\varphi_1(\xi)\,d\xi, \quad \omega_2(x,t) = \frac{1}{\pi}\int\limits_{\mathbb{R}} U(\xi,T;x,t)\varphi_2(\xi)\,d\xi.$$

We use the following integral representation of a solution to the system (1):

$$u^1(x,t) = \int\limits_0^t \overrightarrow{U_1}(x,t;0,\tau)\vec{\alpha}(\tau)\,d\tau + \omega_1(x,t), \quad u^2(x,t) = \int\limits_t^T \overrightarrow{U_2}(0,\tau;x,t)\vec{\beta}(\tau)\,d\tau + \omega_2(x,t), \tag{4}$$

where $\overrightarrow{U_1}, \overrightarrow{U_2}$ are the row vectors $\overrightarrow{U_1} = (U, V_1, \ldots, V_{n-1})$, $\overrightarrow{U_2} = (U, W_1, \ldots, W_{n-1})$, U is a fundamental solution, V_p and W_p are L. Cattabriga's elementary solutions [5] of first equation (1), and $\vec{\alpha}(t)$, $\vec{\beta}(t)$ are column vectors of unknown densities with components $\alpha_p(t), \beta_p(t),\ p = 0, 1, \ldots, n-1$. The functions $\omega_1(x,t)$, $\omega_2(x,t)$ are solutions to the equation (1) satisfying the conditions (2) in $\mathbb{R}$.

Without loss of generality, we can consider the case of the matrix

$$A = \begin{pmatrix} a_{11} & 0 & \ldots & 0 & 0 \\ 0 & -a_{22} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_{2n-2,2n-2} & 1 \\ 0 & 0 & \ldots & 0 & -a_{2n-1,2n-1} \end{pmatrix}.$$

In the case of the symmetric matrix A, we are in the conditions of the article [5]. We assume that the entries a_{ij} of A satisfy the condition of uniqueness of a solution to the boundary value problem (1)–(3), i. e.,

$$\int\limits_0^T \left\{ \left[\sum_{i=0}^{n-1} (-1)^i \frac{\partial^i u^1}{\partial x^i} \cdot \frac{\partial^{2n-1-i} u^1}{\partial x^{2n-1-i}} \right]_{x=+0} - \left[\sum_{i=0}^{n-1} (-1)^i \frac{\partial^i u^2}{\partial x^i} \cdot \frac{\partial^{2n-1-i} u^2}{\partial x^{2n-1-i}} \right]_{x=+0} \right\} dt = 0.$$

As is known, if the densities $\vec{\alpha}(t)$ and $\vec{\beta}(t)$ belong to the space $H^q(0,T)$ ($q = \frac{p-(2n-1)}{2n}$) and satisfy the conditions

$$\vec{\alpha}^{(s)}(0) = \vec{\beta}^{(s)}(T) = 0,\ s = 0, \ldots, l-1, \tag{5}$$

then $u^1(x,t), u^2(x,t) \in H^{p,p/2n}_{x\,t}(Q^+)$, where $p = 2nl + \gamma,\ 0 < \gamma < 1$.

The gluing conditions (3) generate the following system of integral equations with Abel's operators with respect to the unknown densities $\vec{\alpha}(t), \vec{\beta}(t)$:

$$\begin{cases} \int\limits_0^t \dfrac{\partial^i \overrightarrow{U}_1}{\partial x^i}(0,t;0,\tau)\vec{\alpha}(\tau)\,d\tau + \dfrac{\partial^i \omega_1}{\partial x^i}(0,t) = \\ \qquad = (-1)^i a_{ii}\Big[\int\limits_t^T \dfrac{\partial^i \overrightarrow{U}_2}{\partial x^i}(0,\tau;0,t)\vec{\beta}(\tau)\,d\tau + \dfrac{\partial^i \omega_2}{\partial x^i}(0,t)\Big], \\ \qquad\qquad i = 0,1,\ldots,2n-3, \\ \int\limits_0^t \dfrac{\partial^{2n-2} \overrightarrow{U}_1}{\partial x^{2n-2}}(0,t;0,\tau)\vec{\alpha}(\tau)\,d\tau + \dfrac{\partial^{2n-2} \omega_1}{\partial x^{2n-2}}(0,t) = \\ \qquad = a_{2n-2,2n-2}\int\limits_t^T \dfrac{\partial^{2n-2} \overrightarrow{U}_2}{\partial x^{2n-2}}(0,\tau;0,t)\vec{\beta}(\tau)\,d\tau + a_{2n-2,2n-2}\dfrac{\partial^{2n-2} \omega_2}{\partial x^{2n-2}}(0,t) \\ \qquad + \int\limits_{-\infty}^0 \overrightarrow{B}_0(\eta)\,d\eta\cdot\vec{\beta}(t) + (-1)^n \dfrac{2n}{\Gamma\left(\frac{1}{2n}\right)}\dfrac{\partial^{2n-1}\omega_2}{\partial x^{2n-1}}(0,t), \\ \int\limits_0^\infty \overrightarrow{A}_0(\eta)\,d\eta\cdot\vec{\alpha}(t) + (-1)^n \dfrac{2n}{\Gamma\left(\frac{1}{2n}\right)}\dfrac{\partial^{2n-1}\omega_1}{\partial x^{2n-1}}(0,t) = \\ \qquad -a_{2n-1,2n-1}\int\limits_{-\infty}^0 \overrightarrow{B}_0(\eta)\,d\eta\cdot\vec{\beta}(t) - (-1)^n \dfrac{2n a_{2n-1,2n-1}}{\Gamma\left(\frac{1}{2n}\right)}\dfrac{\partial^{2n-1}\omega_2}{\partial x^{2n-1}}(0,t), \end{cases} \tag{6}$$

where

$$\overrightarrow{A}_i(\eta) = \left\{\bar{f}^{(i)}(\eta), \bar{g}_1^{(i)}(\eta), \ldots, \bar{g}_{n-1}^{(i)}(\eta)\right\}, \qquad \overrightarrow{B}_i(\eta) = \left\{\bar{f}^{(i)}(-\eta), \bar{h}_1^{(i)}(-\eta), \ldots, \bar{h}_{n-1}^{(i)}(-\eta)\right\},$$

$$\bar{f}^{(i)}(\eta) = \frac{2n f^{(i)}(\eta)}{\Gamma\left(\frac{1+i}{2n}\right)}, \quad \bar{g}_p^{(i)}(\eta) = \frac{2n g_p^{(i)}(\eta)}{\Gamma\left(\frac{1+i}{2n}\right)}, \quad \bar{h}_p^{(i)}(\eta) = \frac{2n h_p^{(i)}(\eta)}{\Gamma\left(\frac{1+i}{2n}\right)}$$

and the functions $f(\eta)$, $g_p(\eta)$, $h_p(\eta)$ are the solutions to a linear differential equation of $(2n-1)$-th order [5]

$$z^{(2n-1)}(\eta) - \frac{(-1)^n}{2n}\cdot\eta\cdot z(\eta) = 0.$$

Deriving the last equality in (6), we employ the equalities

$$\int\limits_0^t \frac{\partial^{2n-1} V_p(x,t;0,\tau)}{\partial x^{2n-1}}\bigg|_{x=0} \alpha_p(\tau)\,d\tau = (-1)^n \alpha_p(t)\int\limits_0^\infty g_p(\eta)\,d\eta,$$

$$\int\limits_t^T \frac{\partial^{2n-1} W_p(0,\tau;x,t)}{\partial x^{2n-1}}\bigg|_{x=0} \beta_p(\tau)\,d\tau = (-1)^n \beta_p(t)\int\limits_{-\infty}^0 h_p(\eta)\,d\eta.$$

To prove the existence of solutions $\vec{\alpha}(t), \vec{\beta}(t)$ to the system (6) from the space $H^{l-1+\frac{\gamma+1}{2n}}(0,T)$ satisfying the conditions (5), we reduce the boundary value problem (1)–(3) to the following system of singular integral equations of the normal type:

$$K_1\vec{\beta} \equiv A\vec{\beta}(t) - \frac{1}{\pi}\int\limits_0^T \frac{B(t,\tau)\vec{\beta}(\tau)}{\tau - t}\,d\tau = \vec{Q}(t), \tag{7}$$

where A and B are matrices of n-th order written out explicitly. For $n=3$, the matrices A and B are as follows [6]:

$$A=\begin{pmatrix} -a_{66} & -\frac{1}{12}(a_{22}-a_{44}+4a_{66}) & -\frac{\sqrt{3}}{12}a_{22}+a_{44}) \\ 0 & \frac{1}{4}(a_{22}-a_{44}) & \frac{\sqrt{3}}{4}(a_{22}+a_{44}) \\ a_{11} & \frac{1}{2}a_{11}+\frac{\sqrt{3}}{12}(a_{22}+\sqrt{3}a_{44}) & \frac{\sqrt{3}}{2}a_{11}+\frac{1}{4}(a_{22}-a_{44}) \end{pmatrix},$$

$$B\equiv B(t,t)=\begin{pmatrix} \frac{1}{4}a_{33} & -\frac{1}{12}(3a_{33}+\sqrt{3}a_{22}+\sqrt{3}a_{44}) & -\frac{1}{4}(a_{22}-a_{44}) \\ -\frac{3}{4}a_{33} & \frac{\sqrt{3}}{4}(\sqrt{3}a_{33}+a_{22}+a_{44}) & \frac{3}{4}(a_{22}-a_{44}) \\ \frac{\sqrt{3}}{12}(4a_{11}+a_{33}) & \frac{\sqrt{3}}{12}(2a_{11}-a_{33}+\sqrt{3}a_{22}-\sqrt{3}a_{44}) & \frac{\sqrt{3}}{4}(2a_{11}+a_{22}+a_{44}) \end{pmatrix}.$$

Theorem 1 *Let $\varphi_1,\varphi_2\in H^p(\mathbb{R}^+)$ $(p=2nl+\gamma)$, $n\geq 2$ and $\theta\in(\frac{1}{2}-\frac{1}{n};\frac{1}{2}-\frac{1}{2n})$. If the $2nl$ conditions*

$$L_s(\varphi_1,\varphi_2)=0,\quad s=1,\dots,2nl, \tag{8}$$

are fulfilled, then there exists a unique solution to the equation (1) *satisfying the conditions* (2), (3) *from space*

1) $H_{x\ t}^{p,p/2n}(Q^+)$, where $0<\gamma<n-1-2n\theta$;

2) $H_{x\ t}^{q,q/2n}(Q^+)$, $q=2nl+n-1-2n\theta$, where $n-1-2n\theta<\gamma<1$;

3) $H_{x\ \ \ t}^{q-\varepsilon,(q-\varepsilon)/2n}(Q^+)$, if $\gamma=n-1-2n\theta$, where ε is an arbitrarily small positive constant.

Remark 1 *Under the conditions of the theorem for $\theta\leq\frac{1}{2}-\frac{1}{n}$ $(n\geq 3)$, the problem* (1)–(3) *has a unique solution from the space $H_{x\ t}^{p,p/2n}(Q^+)$ under the $2nl$ conditions* (8).

Remark 2 *Under the conditions of the theorem, with $\theta\geq\frac{1}{2}-\frac{1}{2n}$, in accord with [8], the problem* (1)–(3) *has a unique solution from the space $H_{x\ t}^{p,p/2n}(Q^+)$ under $4nl-2l+2$ conditions of form* (8).

PARABOLIC EQUATIONS OF THE FOURTH ORDER

Consider the equation (1) for $n=2$. Let the entries of A be real numbers a_{ij} and

$$A=\begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & -a_{22} & 0 & -a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & -a_{44} \end{pmatrix}. \tag{9}$$

Note that in the case of the symmetric matrix A we are in the conditions of the article [2]. We assume that the entries a_{ij} of the matrix A satisfy the uniqueness condition for solutions to the boundary value problems (1)–(3).

Introduce the notation $\theta=\frac{1}{\pi}\arctan|\frac{a}{b}|$, where $a=a_{33}$ and $b=a_{33}+\sqrt{2}a_{22}$.

Theorem 2 *Assume that $a_{33}+\sqrt{2}a_{22}\neq 0$, $\varphi_1,\varphi_2\in H^p(\mathbb{R}^+)$ $(p=4l+\gamma)$, and $\theta\in(0;\frac{1}{4})$. If $4l$ conditions* (8) *are fulfilled then there exists a unique solution to the equation* (1) *satisfying* (2), (3) *from the space*

1) $H_{x\ t}^{p,p/4}(Q^+)$, if $0<\gamma<1-4\theta$;

2) $H_{x\ t}^{q,q/4}(Q^+)$, $q=4l+1-4\theta$, if $1-4\theta<\gamma<1$;

3) $H_{x\ \ \ t}^{q-\varepsilon,(q-\varepsilon)/4}(Q^+)$, if $\gamma=1-4\theta$, where ε is an arbitrary small positive constant.

Proof. In view of general denseness results, $\alpha_k(t),\beta_k(t)$ $(k=0,1)$ belongs to the space $H^q(0,T)$ $(q=\frac{p-3}{4})$ and

$$\alpha_k^{(s)}(0)=\beta_k^{(s)}(T)=0,\quad s=0,\dots,l-1. \tag{10}$$

The gluing conditions (3) generate the following system of integral equations with Abel's operators with respect to α_k, β_k:

$$\begin{cases} \frac{1}{4}\Gamma(\frac{1}{4})\int\limits_0^t \frac{\alpha_0(\tau)+\alpha_1(\tau)}{(t-\tau)^{\frac{1}{4}}}\,d\tau + \omega_1(0,t) = \frac{a_{11}}{4}\Gamma(\frac{1}{4})\int\limits_t^T \frac{\beta_0(\tau)+\beta_1(\tau)}{(\tau-t)^{\frac{1}{4}}}\,d\tau + a_{11}\omega_2(0,t) + a_{14}(\frac{\pi}{2}\beta_0(t) + \omega_{2xxx}(0,t)), \\ -\frac{1}{2}\Gamma(\frac{1}{2})\int\limits_0^t \frac{\alpha_1(\tau)}{(t-\tau)^{\frac{1}{2}}}\,d\tau - \frac{a_{22}}{2}\Gamma(\frac{1}{2})\int\limits_t^T \frac{\beta_1(\tau)}{(\tau-t)^{\frac{1}{2}}}\,d\tau + \omega_{1x}(0,t) + a_{22}\omega_{2x}(0,t) + a_{24}(\frac{\pi}{2}\beta_0(t) + \omega_{2xxx}(0,t)) = 0, \\ -\frac{1}{4}\Gamma(\frac{3}{4})\int\limits_0^t \frac{\alpha_0(\tau)-\alpha_1(\tau)}{(t-\tau)^{\frac{3}{4}}}\,d\tau + \omega_{1xx}(0,t) = -\frac{a_{33}}{4}\Gamma(\frac{3}{4})\int\limits_t^T \frac{\beta_0(\tau)-\beta_1(\tau)}{(\tau-t)^{\frac{3}{4}}}\,d\tau + a_{33}\omega_{2xx}(0,t) + a_{34}(\frac{\pi}{2}\beta_0(t) + \omega_{2xxx}(0,t)), \\ \frac{\pi}{2}\alpha_0(t) + \omega_{1xxx}(0,t) + a_{44}(\frac{\pi}{2}\beta_0(t) + \omega_{2xxx}(0,t)) = 0. \end{cases} \tag{11}$$

To prove the existence of solutions $\alpha_k(t)$, $\beta_k(t)$ $(k = 0, 1)$ to the system (11) from the space $H^{l-1+\frac{\gamma+1}{4}}(0,T)$ satisfying the conditions (5), the boundary value problem (1)–(3) is reduced to a system of singular integral equations of the normal type of the form

$$K\vec{\beta} \equiv P\vec{\beta}(t) + \frac{1}{\pi}\int\limits_0^T \frac{N(t,\tau)\vec{\beta}(\tau)}{\tau - t}\,d\tau = \vec{Q}(t), \tag{12}$$

where

$$\vec{\beta}(t) = (\tilde{\beta}_0^{(l-1)}(t), \tilde{\beta}_1^{(l-1)}(t)),$$

$$P = \begin{pmatrix} -a_{14}a_{24}a_{33}a_{34} & a_{14}a_{24}a_{33}a_{34} \\ a_{34}a_{44} & 0 \end{pmatrix},$$

$$N(t,\tau) = \begin{pmatrix} a_{14}a_{24}a_{33}a_{34}(\frac{t}{\tau})^{3/4} & a_{14}a_{24}a_{33}a_{34}(\frac{t}{\tau})^{3/4} + \sqrt{2}a_{14}a_{22}a_{24}a_{34}(\frac{t}{\tau})^{1/2} \\ 0 & 0 \end{pmatrix}.$$

It is possible to rewrite the system as follows:

$$\begin{cases} a_{33}\tilde{\beta}_1^{(l-1)}(t) + \frac{a_{33}+\sqrt{2}a_{22}}{\pi}\int\limits_0^T \frac{\tilde{\beta}_1^{(l-1)}(\tau)}{\tau - t}\,d\tau + \frac{1}{\pi}\int\limits_0^T M(t,\tau)\tilde{\beta}_1^{(l-1)}(\tau)\,d\tau = q_1(t), \\ a_{34}a_{44}\tilde{\beta}_0^{(l-1)}(t) = q_2(t). \end{cases} \tag{13}$$

Extract the characteristic part of the singular equation in system (13). We have

$$a\tilde{\beta}_1^{(l-1)}(t) + \frac{b}{\pi}\int\limits_0^T \frac{\tilde{\beta}_1^{(l-1)}(\tau)}{\tau - t}\,d\tau = g(t). \tag{14}$$

A solution of the singular integral equation (14) in the class of functions bounded at the ends of the interval $(0,T)$ is sought using the piecewise-analytic function [7]

$$\Psi(z) = \frac{1}{2\pi i}\int\limits_0^T \frac{\tilde{\beta}_1^{(l-1)}(\tau)}{\tau - z}\,d\tau = \frac{\chi(z)}{2\pi i}\int\limits_0^T \frac{g(\tau)\,d\tau}{(a+bi)\chi^+(\tau)(\tau - z)}, \tag{15}$$

where the canonical function is equal to $\chi(z) = z^{\theta}(z-T)^{1-\theta}$ if a, b have the same sign, and $\chi(z) = z^{1-\theta}(z-T)^{\theta}$ if a, b are of different signs, $\theta = \frac{1}{\pi}\arctan|\frac{a}{b}|$. Since the index of the Riemann boundary value problem is equal to $\varkappa = -1$, the equality (15) is fulfilled under the condition

$$\int_0^T \frac{g(\tau)}{\chi(\tau)}\,d\tau = 0. \tag{16}$$

In this case, we have

$$\tilde{\beta}_1^{(l-1)}(t) = \Psi^+(t) - \Psi^-(t) = \frac{1}{2}g(t) + \frac{\chi(t)}{2\pi}\int_0^T \frac{g(\tau)\,d\tau}{\chi(\tau)(\tau - t)}. \tag{17}$$

The formulas (16) can be treated as necessary and sufficient conditions of boundedness of $\tilde{\beta}_1^{(l-1)}(t)$ at $t = T$.

Substituting $g(t)$ in (17) and taking the first equation in (13) into account, we arrive at the collection of Fredholm equations

$$\vec{\beta} + k\vec{\beta} = \vec{Q}^*, \tag{18}$$

where

$$k\vec{\beta} = \frac{1}{\pi}\int_0^T N(t,\tau)\vec{\beta}(\tau)\,d\tau.$$

Any bounded integrable solution to the system of Fredholm equations (18) belongs to a Hölder space at all points of the contour $(0, T)$ different from the ends. Properties of the kernel $N(t,\tau)$ and a constant term $\vec{Q}^*$ imply that any bounded integrable solution to the system of Fredholm equations (18) at the ends $0, T$ behaves as $t^{\frac{1}{2}+\theta}(T-t)^{\frac{1}{2}-\theta}$ if a and b have the same sign, or as $t^{\frac{1}{2}-\theta}(T-t)^{\frac{1}{2}+\theta}$ if a and b are of different signs.

In view of the lemma about the membership of a Cauchy-type integral in a Hölder class at the ends of an integration contour [9], under the condition $\frac{1+\gamma}{4} < \frac{1}{2} - \theta$ ($\theta < \frac{1}{4}$) the solutions to the Fredholm equations (18) belong to space $H^{\frac{1+\gamma}{4}}(0, T)$ and vanish at $0, T$ with the order $\frac{1+\gamma}{4}$. Moreover, solutions to the Fredholm equations (18) satisfy the Hölder condition with the exponent $\frac{1}{2} - \theta$ for $1 - 4\theta < \gamma < 1$ and with the exponent $\frac{1}{2} - \theta - \varepsilon$ for $\gamma = 1 - 4\theta$.

Thus, the system of equations (18) is equivalent to the initial system of equations (11) under $4l$ conditions of the form (8).

Solvability of the Fredholm system (18) follows from the uniqueness of solutions to the initial problem (1)–(3) and the uniqueness of their representations through potentials. Substitute the Taylor expansion

$$\begin{aligned}&\vec{\beta}^{(s)}(t) = \sum_{k=s}^{l-2} \frac{\vec{\beta}^{(k)}(0)}{(k-s)!}\, t^{k-s} + \frac{1}{(l-2-s)!}\int_0^t (t-\tau)^{l-2-s}\vec{\beta}^{(l-1)}(\tau)\,d\tau,\\ &(s = 0, \ldots, l-2)\end{aligned} \tag{19}$$

into the conditions (8). We obtain $4l$ solvability conditions of problem (1)–(3) in the space $H_{x\ t}^{p,p/4}(Q^+)$, as was to be shown. □

Remark 3 *If the conditions of the theorem are fulfilled for* $\theta \in [\frac{1}{4}, \frac{1}{2})$ *then the problem* (1)–(3) *has a unique solution in the space* $H_{x\ t}^{p,p/4}(Q^+)$ *under* $6l + 2$ *conditions of the form* (8) [8].

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