On the Problem of Recovering the Coefficients in a One-Dimensional Third Order Equation

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Abstract. We prove the existence and uniqueness of solutions to inverse problems for third-order equations with the pointwise overdetermination. The unknowns are a solution to the equation and two or three external sources.

INTRODUCTION

The problems of recovering coefficients of partial differential equations and systems with given additional information about a solution are of great practical importance [1–3]. Note that the inverse problems for hyperbolic equations often are regarded as ill-posed problems of mathematical physics whose theory was developed in articles by A. N. Tikhonov [4–6], V. K. Ivanov [7], and M. M. Lavrent'ev [8,9].

The problems of recovering densities of external sources often arise in the theory of inverse problems of heat and mass transfer [10–12]. It is often the case when the unknown right-hand side depends on time [13] and inverse problems are posed as control problems [14]. The monographs [15, 16] are devoted to the study of inverse problems for higher order parabolic equations. Observe that the spatially nonlocal direct problems for third order equations are well studied (see, for instance, [17–19]) in contrast to the inverse problems for equations of this type. The unknown parameter depending on time is examined in [20, 21] for parabolic equations and in [22–24] for hyperbolic.

In this article we establish the solvability of an inverse problem of recovering external sources together with a solution for a third order equation in time with point overdetermination conditions; the densities of two or three sources are recovered.

STATEMENTS OF INVERSE PROBLEMS

Let Ω be the interval $(0, 1)$ of the Ox-axis and Q the rectangle $\Omega \times (0, T)$ with $0 < T < +\infty$.

Problem 1 *Find $u(x, t)$, $q_1(t)$, and $q_2(t)$ satisfying the equation*

$$u_{ttt} + u_{xx} + c(x, t)u = f(x, t) + q_1(t)h_1(x, t) + q_2(t)h_2(x, t) \tag{1}$$

in Q, the initial conditions

$$u(x, 0) = u_t(x, 0) = u(x, T) = 0, \quad x \in \Omega, \tag{2}$$

the boundary conditions

$$u_x(0, t) = 0, \quad u_x(1, t) = 0, \quad t \in (0, T), \tag{3}$$

and the overdetermination conditions

$$u(0, t) = 0, \quad u(1, t) = 0, \quad t \in (0, T). \tag{4}$$

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Problem 2 *Find $u(x,t)$, $q_1(t)$, $q_2(t)$ and $q_3(t)$ satisfying the equation*

$$u_{ttt} + u_{xx} + c(x,t)u = f(x,t) + q_1(t)h_1(x,t) + q_2(t)h_2(x,t) + q_3(t)h_3(x,t) \tag{5}$$

in Q, the initial conditions (2)*, the boundary conditions* (3)*, and the overdetermination conditions*

$$u(0,t) = 0, \quad u(\alpha,t) = 0, \quad u(1,t) = 0, \quad \alpha \in (0,1), \quad t \in (0,T). \tag{6}$$

In [22], the Fourier method is applied to study an inverse problem of recovering densities of sources in the one-dimensional wave equation with constants coefficients. In work [25] the density of an external source is recovered in the case of a single source.

SOLVABILITY OF BOUNDARY VALUE PROBLEM 1

Put:

$$\triangle(t) = h_1(0,t)h_2(1,t) - h_2(0,t)h_1(1,t),$$

$$\widetilde{f_1}(x,t) = \frac{h_1(x,t)}{\triangle(t)} \cdot (h_2(0,t)f(1,t) - h_2(1,t)f(0,t)) - \frac{h_2(x,t)}{\triangle(t)} \cdot (h_1(0,t)f(1,t) - h_1(1,t)f(0,t)) + f(x,t),$$

$$\alpha_0(t) = \widetilde{f}_{1x}(0,t), \quad \beta_0(t) = \widetilde{f}_{1x}(1,t),$$

$$\alpha_1(t) = \frac{1}{\triangle(t)}(h_2(1,t)h_{1\,x}(0,t) - h_1(1,t)h_{2\,x}(0,t)), \quad \alpha_2(t) = \frac{1}{\triangle(t)}(h_1(0,t)h_{2\,x}(0,t) - h_2(0,t)h_{1\,x}(0,t)),$$

$$\beta_1(t) = \frac{1}{\triangle(t)}(h_2(1,t)h_{1\,x}(1,t) - h_1(1,t)h_{2\,x}(1,t)), \quad \beta_2(t) = \frac{1}{\triangle(t)}(h_1(0,t)h_{2\,x}(1,t) - h_2(0,t)h_{1\,x}(1,t)),$$

$$\alpha(x,t) = \frac{1}{\triangle(t)}(h_2(1,t)h_1(x,t) - h_1(1,t)h_2(x,t)), \quad \beta(x,t) = \frac{1}{\triangle(t)}(h_1(0,t)h_2(x,t) - h_2(0,t)h_1(0,t).$$

Introduce the notation

$$h_1 = \max_{\overline{Q}}(|\alpha_{xx}|, |\beta_{xx}|). \tag{7}$$

Denote by $V_0 = W^{2,3}_{2,x,t}(Q)$ an anisotropic Sobolev space and by W_0, W_1 the vector spaces

$$W_0 = \{v(x,t) : \ v(x,t) \in V_0, \ v_{xxttt} \in L_2(Q)\}, \quad W_1 = \{v(x,t) : \ v(x,t) \in W_0, \ v_x \in W_0\}$$

endowed with the norms

$$\|v\|_{W_0} = \|v\|_{V_0} + \|v_{xxttt}\|_{L_2(Q)}, \quad \|v\|_{W_1} = \|v\|_{W_0} + \|v_x\|_{W_0}.$$

Before proving the solvability of Boundary Value Problem 1, we observe that for a function $v(x,t)$ from V_0 satisfying (2), the following inequalities hold:

$$v^2(0,t) + v^2(1,t) \leq \delta_1 \int_0^1 v_x^2(x,t)dx + C_1(\delta_1) \int_0^1 v^2(x,t)dx, \tag{8}$$

$$\int_0^T v_t^2(x,t)dt \leq \delta_2 \int_0^T v_{ttt}^2(x,t)dt + C_2(\delta_2,T) \int_0^T v^2(x,t)dt, \tag{9}$$

$$\int_0^T v_{tt}^2(x,t)dt \leq \delta_3 \int_0^T v_{ttt}^2(x,t)dt + C_3(\delta_3,T) \int_0^T v^2(x,t)dt, \tag{10}$$

where δ_1, δ_2, and δ_3 are arbitrary positive numbers and C_1, C_2, and C_3 are calculated through δ_1, δ_2, δ_3, and T.

Theorem 1 *Assume that*

$$c(t) \in C^1[0,T], \quad -c(t) \geq c_0 \gg 0 \quad for \quad t \in [0,T],$$
$$h_i(x,t) \in C^3(\overline{Q}), \quad h_1 < \tfrac{1}{2T}, \quad \triangle(t) \neq 0, \tag{11}$$
$$\alpha_2(t) + \beta_2(t) < 2 \quad for \quad t \in [0,T],$$

$$\alpha_1(t)\xi_1^2 + [-\beta_1(t) + \alpha_2(t)]\,\xi_1\xi_2 - \beta_2(t)\xi_2^2 \geq 0 \quad for \quad t \in [0,T], \quad (\xi_1,\xi_2) \in \mathbb{R}^2, \tag{12}$$

$$\alpha_1^2(t) + \beta_1^2(t) < \tfrac{1}{2},$$
$$\tfrac{1}{2} + 2(\alpha_2(t)\beta_1(t) - \alpha_1(t)\beta_2(t)) \geq \alpha_1^2(t) + \beta_1^2(t) + \alpha_2^2(t) + \beta_2^2(t) \quad for \quad t \in [0,T], \tag{13}$$

$$f(x,t) \in W_2^3(Q), \quad f_{xxttt}(x,t) \in L_2(Q). \tag{14}$$

Then there exists a regular solution to the problem (1)–(4) *such that* $u(x,t)$ *and* $u_{xx}(x,t)$ *belong to* $W^{2,3}_{2,x,t}(Q)$ *and* $q_1(t), q_2(t) \in L_2(0,T)$.

Proof. Consider the auxiliary boundary value problem: find a solution $u(x,t)$ to the equation

$$u_{ttt} + u_{xx} + c(t)u = \widetilde{f}_{1xx}(x,t) + \lambda[\alpha_{xx}(x,t)u(0,t) + \beta_{xx}(x,t)u(1,t)] \tag{15}$$

in Q such that the nonlocal conditions

$$u_x(0,t) = \alpha_1(t)u(0,t) + \alpha_2(t)u(1,t) + \alpha_0(t), \quad 0 < t < T,$$
$$u_x(1,t) = \beta_1(t)u(0,t) + \beta_2(t)u(1,t) + \beta_0(t), \quad 0 < t < T \tag{16}$$

and the initial conditions

$$u(x,0) = u_t(x,0) = u(x,T) = 0, \quad x \in \Omega, \tag{17}$$

hold, where λ is some real parameter defined below. Note that spatially nonlocal boundary value problems of the form (15)–(17) for nonloaded equation (15) are considered in [26, 27].

Given $(x,t) \in \overline{Q}$, put

$$\gamma(x,t) = \frac{x^2}{2}[\beta_0(t) - \alpha_0(t)] + x\alpha_0(t), \quad v(x,t) = u(x,t) - \gamma(x,t).$$

In this case, instead of (15)–(17) we can consider the following boundary value problem: find a solution $v(x,t)$ to the equation

$$v_{ttt} + v_{xx} + c(t)v = f_2(x,t) + \lambda[\alpha_{xx}(x,t)v(0,t) + \beta_{xx}(x,t)v(1,t)] \tag{18}$$

in Q such that

$$v_x(0,t) = \alpha_1(t)v(0,t) + \alpha_2(t)v(1,t), \quad 0 < t < T,$$
$$v_x(1,t) = \beta_1(t)v(0,t) + \beta_2(t)v(1,t), \quad 0 < t < T, \tag{19}$$

$$v(x,0) = v_t(x,0) = v(x,T) = 0, \quad x \in \Omega, \tag{20}$$

where

$$f_2(x,t) = \widetilde{f}_{1xx}(x,t) + B_0(x,t), \quad B_0(x,t) = \tfrac{1}{2}\lambda\beta(x,t)(\alpha_0(t) + \beta_0(t)) - \gamma_{ttt}(x,t) - \gamma_{xx}(x,t) - c(t)\gamma(x,t).$$

Without loss of generality, we can consider the homogeneous initial conditions (20) under the following assumptions:

$$\widetilde{f}_{1x}(0,0) = \widetilde{f}_{1xt}(0,0) = \widetilde{f}_{1x}(0,T) = 0, \quad \widetilde{f}_{1x}(1,0) = \widetilde{f}_{1xt}(1,0) = \widetilde{f}_{1x}(1,T) = 0.$$

Given $(x,t) \in \overline{Q}$, $\lambda \in [0,1]$, put

$$\gamma_1(x,t,\lambda) = \tfrac{\lambda x^2}{2}[\beta_1(t) - \alpha_1(t)] + \lambda x\alpha_1(t), \quad \delta_1(x,t,\lambda) = \tfrac{\lambda x^2}{2}[\beta_2(t) - \alpha_2(t)] + \lambda x\alpha_2(t),$$
$$w(x,t) = v(x,t) - \gamma_1(x,t,\lambda)v(0,t) - \delta_1(x,t,\lambda)v(1,t),$$
$$\gamma_{11}(x,t,\lambda) = \gamma_1(x,t,\lambda) + \tfrac{\delta_1(x,t,\lambda)\gamma_1(1,t,\lambda)}{1-\delta_1(1,t,\lambda)}, \quad \delta_{11}(x,t,\lambda) = \tfrac{\delta_1(x,t,\lambda)}{1-\delta_1(1,t,\lambda)}$$
$$v(x,t) = w(x,t) + \gamma_{11}(x,t,\lambda)w(0,t) + \delta_{11}(x,t,\lambda)w(1,t).$$

Let $v(x,t)$ be a solution to (18). In this case $w(x,t)$ satisfies the equality

$$w_{ttt} + w_{xx} + c(t)w = f_2(x,t) + \Phi(x,t,\lambda,\overline{w}(t)),$$

where

$$\begin{gathered}
\overline{w}(t) = (w_{ttt}(0,t), w_{ttt}(1,t), w_{tt}(0,t), w_{tt}(1,t), w_t(0,t), w_t(1,t), w(0,t), w(1,t)),\\
\Phi(x,t,\lambda,\overline{w}(t)) = A_1(x,t,\lambda)w_{ttt}(0,t) + A_2(x,t,\lambda)w_{ttt}(1,t) + A_3(x,t,\lambda)w_{tt}(0,t) + A_4(x,t,\lambda)w_{tt}(1,t)+\\
+A_5(x,t,\lambda)w_t(0,t) + A_6(x,t,\lambda)w_t(1,t) + A_7(x,t,\lambda)w(0,t) + A_8(x,t,\lambda)w(1,t),\\
A_1(x,t,\lambda) = -\gamma_{11}(x,t,\lambda),\ \ A_2(x,t,\lambda) = -\delta_{11}(x,t,\lambda),\ \ A_3(x,t,\lambda) = -3\gamma_{11\,t}(x,t,\lambda),\\
A_4(x,t,\lambda) = -3\delta_{11\,t}(x,t,\lambda),\ \ A_5(x,t,\lambda) = -3\gamma_{11\,tt}(x,t,\lambda),\ \ A_6(x,t,\lambda) = -3\delta_{11\,tt}(x,t,\lambda),\\
A_7(x,t,\lambda) = \lambda\alpha - \gamma_{11\,ttt}(x,t,\lambda) - \gamma_{11\,xx}(x,t,\lambda) - c(t)\gamma_{11}(x,t,\lambda),\\
A_8(x,t,\lambda) = \lambda\beta - \delta_{11\,ttt}(x,t,\lambda) - \delta_{11\,xx}(x,t,\lambda) - c(t)\delta_{11}(x,t,\lambda).
\end{gathered}$$

Consider the auxiliary boundary value problem: find a solution $w(x,t)$ to the eqaution

$$w_{ttt} + w_{xx} + c(t)w = f_2(x,t) + \Phi(x,t,\lambda,\overline{w}(t)) \tag{21}$$

in the rectangle Q such that

$$\begin{aligned}
&w_x(0,t) = w_x(1,t) = 0,\ \ t \in (0,T),\\
&w(x,0) = w_t(x,0) = w(x,T) = 0,\ \ x \in \Omega.
\end{aligned} \tag{22}$$

Prove that this problem is solvable in V_0. To this end, we employ the methods of continuation in a parameter and regularization.

Let ε be a positive number; without loss of generality, we can assume that $0 < \varepsilon < 1$. Examine the new boundary value problem: find a solution $w(x,t)$ to the equation

$$L_\varepsilon(\lambda)w \equiv w_{ttt} + w_{xx} + c(t)w - \varepsilon w_{xxttt} = f_2(x,t) + \Phi(x,t,\lambda,\overline{w}(t)) \tag{23}$$

in Q such that (22) hold. Demonstrate that the boundary value problem (23), (22) for a fixed parameter $\varepsilon > 0$ is solvable in W_1 for every $f_2(x,t) \in L_2(Q)$ such that $f_{2x}(x,t) \in L_2(Q)$.

In accord with the method of continuation in a parameter [28], for the boundary value problem (23), (22) to be solvable in the space W_1 for all $\lambda \in [0,1]$ and every $f(x,t)$ from $W^{1,0}_{2,x,t}(Q)$, it is sufficient to establish

1) the continuity of the family of operators $\{L_\varepsilon(\lambda)\}$ in λ;

2) solvability of the boundary value problem (23), (22) for $\lambda = 0$;

3) an a priori estimate in W_0 uniform in λ for solutions $v(x,t)$ to (23), (22).

The continuity in λ of the family $\{L_\varepsilon(\lambda)\}$ of operators is obvious. For $\lambda = 0$ and a fixed number ε, the boundary value problem (23), (22) is solvable in W_1 (see [29]) under the conditions of Theorem 1. Justify an a priori estimate in the space W_1 uniform in λ for all solutions $w(x,t)$ (23), (22).

Let $w(x,t)$ be a solution to (23), (22) from W_1. Put $v(x,t) = w(x,t) + \gamma_{11}(x,t,\lambda)w(0,t) + \delta_{11}(x,t,\lambda)w(1,t)$. The inequality

$$\int_0^T (v^2(0,t) + v^2(1,t))\,dt \le 2\int_0^T\int_0^1 v_x^2(x,t)dxdt + 4\int_0^T\int_0^1 v^2(x,t)dxdt$$

easily implies that $v(x,t)$ belongs to W_1 too and is a solution to the boundary value problem

$$L_\varepsilon(\lambda)v \equiv v_{ttt} + v_{xx} + c(t)v - \varepsilon v_{xxttt} = f_2(x,t) + \lambda\alpha_{xx}(x,t)v(0,t) + \lambda\beta_{xx}(x,t)v(1,t), \tag{24}$$

$$v_x(0,t) = \lambda[\alpha_1(t)v(0,t) + \alpha_2(t)v(1,t)],\quad v_x(1,t) = \lambda[\beta_1(t)v(0,t) + \beta_2(t)v(1,t)],\quad 0 < t < T, \tag{25}$$

$$v(x,0) = v_t(x,0) = v(x,T) = 0,\quad x \in \Omega. \tag{26}$$

Consider the equality

$$\int_0^T\int_0^1 L_\varepsilon(\lambda)v\cdot[-v + v_{ttt}](\lambda_0 - t)\,dxdt = \int_0^T\int_0^1 [f_2(x,t) + \lambda\alpha_{xx}(x,t)v(0,t) + \lambda\beta_{xx}(x,t)v(1,t)][-v + v_{ttt}](\lambda_0 - t)\,dxdt.$$

Integrating by parts and accounting for the initial-boundary conditions (25), (26) on $v(x,t)$, we arrive at the equality

$$\int_0^T\int_0^1[v_x^2 - c(t)v^2](\lambda_0 - t)dxdt + \frac{3}{2}\int_0^T\int_0^1 v_t^2\,dxdt + \frac{1}{2}\int_0^1 v_t^2(x,T)(\lambda_0 - T)\,dx +$$
$$+\frac{3(1+\varepsilon)}{2}\int_0^T\int_0^1 v_{xt}^2 dxdt + \frac{1+\varepsilon}{2}\int_0^1 v_{xt}^2(x,T)(\lambda_0 - T)\,dx + \int_0^T\int_0^1 v_{ttt}^2(\lambda_0 - t)\,dxdt + \varepsilon\int_0^T\int_0^1 v_{xttt}^2(\lambda_0 - t)\,dxdt +$$
$$+\varepsilon\lambda\int_0^T [\alpha_1(t)v_{ttt}^2(0,t) + (\alpha_2(t) - \beta_1(t))v_{ttt}(0,t)v_{ttt}(1,t) - \beta_2(t)v_{ttt}^2(1,t)](\lambda_0 - t)\,dt = -\int_0^T\int_0^1 c(t)vv_{ttt}(\lambda_0 - t)\,dxdt +$$
$$+\varepsilon\lambda\int_0^T\{[3\beta_1'(t)v_{tt}(0,t) + 3\beta_1''(t)v_t(0,t) + \beta_1'''(t)v(0,t) + 3\beta_2'(t)v_{tt}(1,t) + 3\beta_2''(t)v_t(1,t) + \beta_2'''(t)v(1,t)]v_{ttt}(1,t) -$$
$$-[3\alpha_1'(t)v_{tt}(0,t) + 3\alpha_1''(t)v_t(0,t) + \alpha_1'''(t)v(0,t) + 3\alpha_2'(t)v_{tt}(1,t) + 3\alpha_2''(t)v_t(1,t) + \alpha_2'''(t)v(1,t)]v_{ttt}(0,t)\}(\lambda_0 - t)\,dt +$$
$$+\varepsilon\int_0^T\{[(v_{xtt}v_t)(1,t) - (v_{xtt}v_t)(0,t)](\lambda_0 - t) + (v_{xt}v_t)(1,t) - (v_{xt}v_t)(0,t)\}\,dt +$$
$$+\int_0^T\{[(v_xv + v_{xt}v_{tt})(1,t) - (v_xv + v_{xt}v_{tt})(0,t)](\lambda_0 - t) + v_{xt}v_t(1,t) - v_{xt}v_t(0,t)\}\,dt +$$
$$+\int_0^T\int_0^1[f_2(x,t) + \lambda\alpha_{xx}(x,t)v(0,t) + \lambda\beta_{xx}(x,t)v(1,t)](-v + v_{ttt})(\lambda_0 - t)\,dxdt.$$

In view (11), (12) for $\lambda_0 = 2T$, the Young inequality and (8)–(10) yield

$$\int_0^T\int_0^1[v_x^2 - c(t)v^2]dxdt + \int_0^T\int_0^1 v_t^2\,dxdt + \int_0^1 v_t^2(x,T)\,dx + (1+\varepsilon)\left[\int_0^T\int_0^1 v_{xt}^2 dxdt + \int_0^1 v_{xt}^2(x,T)\,dx\right] +$$
$$+\int_0^T\int_0^1 v_{ttt}^2\,dxdt + \varepsilon\int_0^T\int_0^1 v_{xttt}^2\,dxdt \le 2T\delta_0\int_0^T\int_0^1(v^2 + v_{ttt}^2)\,dxdt + \frac{T}{\delta_0}\int_0^T\int_0^1 f_2^2 dxdt +$$
$$+2Th_1\left[\int_0^T(v^2(0,t) + v^2(1,t))\,dt + \int_0^T\int_0^1(v^2 + v_{ttt}^2)\,dxdt\right],$$

hence

$$\int_0^T\int_0^1[v_x^2 - c(t)v^2]dxdt + \int_0^T\int_0^1 v_t^2\,dxdt + \int_0^1 v_t^2(x,T)\,dx + (1+\varepsilon)\left[\int_0^T\int_0^1 v_{xt}^2 dxdt + \int_0^1 v_{xt}^2(x,T)\,dx\right] +$$
$$+\int_0^T\int_0^1 v_{ttt}^2\,dxdt + \varepsilon\int_0^T\int_0^1 v_{xttt}^2\,dxdt \le 2T\delta_0\int_0^T\int_0^1(v^2 + v_{ttt}^2)\,dxdt + \frac{T}{\delta_0}\int_0^T\int_0^1 f_2^2 dxdt + \quad (27)$$
$$+4Th_1\left[\delta_1\int_0^T\int_0^1 v_x^2\,dxdt + C(\delta_1)\int_0^T\int_0^1 v^2\,dxdt\right] + Th_1\int_0^T\int_0^1(v^2 + v_{ttt}^2)\,dxdt,$$

where δ_1 is an arbitrary positive number satisfying the inequality

$$1 - 4Th_1\delta_1 > 0.$$

For a fixed δ_1 and $\delta_0 = \frac{1}{4T}$, in view of the conditions of Theorem 1 (11) there exists a sufficiently large $-c_0 > 0$ such that

$$-c_0 - 2T\delta_0 - 4Th_1C(\delta_1) - Th_1 > 0.$$

Therefore, (27) implies that

$$\int_0^T\int_0^1[v_x^2 - c(t)v^2]dxdt + \int_0^T\int_0^1 v_t^2\,dxdt + \int_0^1 v_t^2(x,T)\,dx + (1+\varepsilon)\left[\int_0^T\int_0^1 v_{xt}^2 dxdt + \int_0^1 v_{xt}^2(x,T)\,dx\right] + \quad (28)$$
$$+\int_0^T\int_0^1 v_{ttt}^2\,dxdt + \varepsilon\int_0^T\int_0^1 v_{xttt}^2\,dxdt \le M_1\int_0^T\int_0^1 f_2^2\,dxdt$$

where the constant M_1 is determined by c_0, $f(x,t)$, $h_1(x,t)$, and $h_2(x,t)$.

Consider the equality

$$\int_0^T L_\varepsilon(\lambda)v\cdot[-v_{xxttt}+(x-\tfrac{1}{2})v_{xttt}+v_{xx}+v_{ttt}]dxdt=\int_0^T\int_0^1 F\cdot[-v_{xxttt}+(x-\tfrac{1}{2})v_{xttt}+v_{xx}+v_{ttt}]dxdt,$$
$$F=f_2(x,t)+\lambda[\alpha_{xx}(x,t)v(0,t)+\beta_{xx}(x,t)v(1,t)].$$

Integrating by parts, using the initial-boundary conditions (25), (26) for $v(x,t)$, we obtain the equality

$$\begin{aligned}
&(1+\tfrac{3\varepsilon}{2})\int_0^T\int_0^1 v_{xttt}^2\,dxdt+\int_0^T\int_0^1[v_{xx}^2+\tfrac{1}{2}v_{ttt}^2]dxdt+\int_0^1 v_{xt}^2(x,T)dx+\varepsilon\int_0^T\int_0^1 v_{xxttt}^2\,dxdt+\tfrac{1+\varepsilon}{2}\int_0^1 v_{xxt}^2(x,T)\,dx+\\
&+\int_0^T\{[\lambda(1+\varepsilon)\alpha_1(t)+\tfrac{1}{4}-\tfrac{\lambda\varepsilon}{4}(\alpha_1^2(t)+\beta_1^2(t))]v_{ttt}^2(0,t)+[\lambda(1+\varepsilon)(\alpha_2(t)-\beta_1(t))-\tfrac{1}{2}\varepsilon\lambda(\alpha_1(t)\alpha_2(t)+\\
&\quad+\beta_1(t)\beta_2(t))]v_{ttt}(0,t)v_{ttt}(1,t)+[\tfrac{1}{4}-\lambda(1+\varepsilon)\beta_2(t)-\tfrac{1}{4}\varepsilon\lambda(\alpha_2^2(t)+\beta_2^2(t)]v_{ttt}^2(1,t)\}dt=\\
&=\tfrac{\lambda\varepsilon}{4}\int_0^T\{[\sum_{k=0}^{2}C_3^k(\alpha_1^{(3-k)}(t)v^{(k)}(0,t)+\alpha_2^{(3-k)}(t)v^{(k)}(1,t))]^2+[\sum_{k=0}^{2}C_3^k(\beta_1^{(3-k)}(t)v^{(k)}(0,t)+\beta_2^{(3-k)}(t)v^{(k)}(1,t))]^2\}\,dt+\\
&+\int_0^T\int_0^1[-c(t)v_xv_{xttt}-(x-\tfrac{1}{2})v_{xx}v_{xttt}-(x-\tfrac{1}{2})c(t)vv_{xttt}-c(t)vv_{xx}-c(t)vv_{ttt}+F_xv_{xttt}+(x-\tfrac{1}{2})Fv_{xttt}+Fv_{xx}+\\
&\quad+Fv_{ttt}]dxdt-\lambda(1+\varepsilon)\int_0^T\{\sum_{k=0}^{2}C_3^k[(\alpha_1^{(3-k)}(t)v^{(k)}(0,t)+\alpha_2^{(3-k)}(t)v^{(k)}(1,t))v_{ttt}(0,t)-(\beta_1^{(3-k)}(t)v^{(k)}(0,t)+\\
&\quad+\beta_2^{(3-k)}(t)v^{(k)}(1,t))v_{ttt}(1,t)]\}dt+\int_0^T[c(t)v(1,t)v_{xttt}(1,t)-c(t)v(0,t)v_{xttt}(0,t)-2v_{xt}(0,t)v_{tt}(0,t)+\\
&\quad+2v_{xt}(1,t)v_{tt}(1,t)+F(0,t)v_{xttt}(0,t)-F(1,t)v_{xttt}(1,t)]dt.
\end{aligned}$$

Using (11)–(13), replacing $v_{xttt}(0,t)$, $v_{xttt}(1,t)$ from (25) and estimating the summands on the right-hand side with the help of Young inequality, (8) and (28), we obtain the a priori estimate

$$\begin{aligned}
&(1+\varepsilon)\int_0^T\int_0^1 v_{xttt}^2\,dxdt+\int_0^T\int_0^1[v_{xx}^2+v_{ttt}^2]dxdt+\int_0^1 v_{xt}^2(x,T)dx+\\
&+\varepsilon\int_0^T\int_0^1 v_{xxttt}^2\,dxdt+(1+\varepsilon)\int_0^1 v_{xxt}^2(x,T)\,dx\le M_2\int_0^T\int_0^1[f_2^2+f_{2x}^2]\,dxdt
\end{aligned}\tag{29}$$

with the constant M_2 defined by c_0, $f(x,t)$, $h_1(x,t)$, and $h_2(x,t)$.

Consider the equality

$$\int_0^T L_\varepsilon(\lambda)v_x\cdot v_{xxx}\,dxdt=\int_0^T\int_0^1[f_{2x}(x,t)+\lambda\alpha_{xx}(x,t)v(0,t)+\lambda\beta_{xx}(x,t)v(1,t)]\cdot v_{xxx}\,dxdt.$$

Integrating by parts and using the Young inequality, (28), (29), and the initial conditions (26) for $v(x,t)$, we arrive at the equality

$$\int_0^T\int_0^1 v_{xxx}^2\,dxdt+\varepsilon\int_0^1 v_{xxxt}^2(x,T)\,dxdt\le M_3\int_0^T\int_0^1 f_{2x}^2\,dxdt\tag{30}$$

with the constant M_3 determined by c_0, $f(x,t)$, $h_1(x,t)$, and $h_2(x,t)$.

Estimates (28)–(30) and equation (24) yield the estimate

$$\|v\|_{W_1}\le M_0$$

uniform in λ.

These estimates allows us to apply the method of continuation in a parameter. Hence, under the conditions of the theorem, the boundary value problem (24)–(26) has a solution $v(x,t)=v_\varepsilon(x,t)$ from W_1 for all values λ including $\lambda=1$. Demonstrate that the family of solutions $\{v_\varepsilon(x,t)\}$ satisfies an a priori estimate uniform in ε which allows us to pass to the limit as $\varepsilon\to 0$.

The solutions $\{v_\varepsilon(x,t)\}$ to (24)–(26) satisfy (28)–(30). Choose a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to \infty$. By the theorem on weak closedness of a bounded set in $L_2(Q)$, there exists a sequence $\{v_m(x,t)\}$ and function $v(x,t)$ such that $v_m(x,t) \to v(x,t)$ weakly in $W^{2,3}_{2,x,t}(Q)$, $v_{m\,x}(x,t) \to v_x(x,t)$ weakly in $W^{2,3}_{2,x,t}(Q)$, $\varepsilon_m w_{m\,xxttt}(x,t) \to 0$ weakly in $L_2(Q)$ as $m \to \infty$. It is obvious that the limit function $v(x,t)$ satisfies (18). Put $w(x,t) = v(x,t) - \gamma_1(x,t,1)v(0,t) - \delta_1(x,t,1)v(1,t)$, then the function $w(x,t)$ belongs to V_0 and it is a solution to (18)–(20).

It remains to show that a solution $u(x,t)$, $q_1(t)$, $q_2(t)$ to the boundary value problem (1)–(4) is determined by $v(x,t)$. Indeed,

$$u_{xx}(x,t) = v(x,t), \quad u_x(0,t) = u_x(1,t) = 0.$$

The function $u(x,t)$ can be determined from these equalities. Let

$$w(x,t) = u_{ttt} + u_{xx} + c(t)u - \widetilde{f_1}(x,t) - \alpha(x,t)v(0,t) - \beta(x,t)v(1,t).$$

In this case (15)–(17) imply that $w(x,t)$ satisfies the equalities

$$w_{xx}(x,t) = 0, \quad w_x(0,t) = w_x(1,t) = 0$$

and so $w(x,t) \equiv 0$ for all $t \in [0,T]$.

Thus, $u(x,t)$ is a solution to the equation

$$u_{ttt} + u_{xx} + c(t)u - \widetilde{f_1}(x,t) - \alpha(x,t)v(0,t) - \beta(x,t)v(1,t). \tag{31}$$

Taking $x = 0$ and $x = 1$ in (31), we conclude that

$$u_{ttt}(0,t) + u_{xx}(0,t) + c(t)u(0,t) - \widetilde{f_1}(0,t) - \alpha(0,t)v(0,t) - \beta(0,t)v(1,t), \tag{32}$$

$$u_{ttt}(1,t) + u_{xx}(1,t) + c(t)u(1,t) - \widetilde{f_1}(1,t) - \alpha(1,t)v(0,t) - \beta(1,t)v(1,t). \tag{33}$$

The boundary value problems

$$\begin{aligned} u_{ttt}(0,t) + c(t)u(0,t) = 0, \quad u(0,0) = u_t(0,0) = u(0,T) = 0,\\ u_{ttt}(1,t) + c(t)u(1,t) = 0, \quad u(1,0) = u_t(1,0) = u(1,T) = 0 \end{aligned}$$

have only zero solutions; i.e., $u(0,t) \equiv 0$ and $u(1,t) \equiv 0$.

Taking the conditions $u(0,t) = 0$, $u(1,t) = 0$ and (32), (33) into account, we derive that $u_{xx}(0,t) = v(0,t)$, $u_{xx}(1,t) = v(1,t)$, i.e., $u(x,t)$ meets (2)–(4). Hence, our function $u(x,t)$ and

$$q_1(t) = \frac{1}{\triangle(t)}[h_2(1,t)(u_{xx}(0,t) - f(0,t)) - h_2(0,t)(u_{xx}(1,t) - f(1,t))],$$

$$q_2(t) = \frac{1}{\triangle(t)}[h_1(0,t)(u_{xx}(1,t) - f(1,t)) - h_1(1,t)(u_{xx}(0,t) - f(0,t))]$$

belongs to the required classes, satisfies (1), and defines a solution to Inverse Boundary Value Problem 1. □

SOLVABILITY OF BOUNDARY VALUE PROBLEM 2

For $(x,t) \in \overline{Q}$ introduce the notations

$$A(t) = \begin{pmatrix} h_1(0,t) & h_2(0,t) & h_3(0,t) \\ h_1(\alpha,t) & h_2(\alpha,t) & h_3(\alpha,t) \\ h_1(1,t) & h_2(1,t) & h_3(1,t) \end{pmatrix}, \quad \triangle(t) = \det|A(t)|,$$

$$\widehat{f_1}(x,t) = -\frac{1}{\triangle(t)}\left[h_1(x,t)\cdot\triangle_1(t) + h_2(x,t)\cdot\triangle_2(t) + h_3(x,t)\cdot\triangle_3(t)\right] + f(x,t),$$

$$\alpha(x,t)=\frac{\triangle^1(x,t)}{\triangle(t)},\quad \beta(x,t)=\frac{\triangle^2(x,t)}{\triangle(t)},\quad \delta(x,t)=\frac{\triangle^3(x,t)}{\triangle(t)},$$
$$\alpha_0(t)=\widehat{f}_{1x}(0,t),\quad \beta_0(t)=\widehat{f}_{1x}(1,t),$$
$$\alpha_1(t)=\alpha_x(0,t),\quad \alpha_2(t)=\beta_x(0,t),\quad \alpha_3(t)=\delta_x(0,t),$$
$$\beta_1(t)=\alpha_x(1,t),\quad \beta_2(t)=\beta_x(1,t),\quad \beta_3(t)=\delta_x(1,t),$$

where $\triangle_i(t)$ are determinants obtained from the determinant $\triangle(t)$ replacing the i-th column with the column vector $\overrightarrow{F}(t)=(f(0,t),f(\alpha,t),f(1,t))$ and $\triangle^i(x,t)$ are determinants obtained from the determinant $\triangle(t)$ replacing the i-th row with the row vector $\overrightarrow{H}(x,t)=(h_1(x,t),h_2(x,t),h_3(x,t))$.

We assume that one of the following three conditions hold:

$$\begin{gathered} p_1(t)+p_2(t)<1,\quad p_2^2(t)-4p_1(t)<0,\quad \text{or} \\ p_1(t)+p_2(t)<1,\quad p_1(t)<0,\quad \text{or} \\ p_1(t)=0,\quad p_2(t)<1, \end{gathered} \tag{34}$$

where

$$p_1(t)=\tfrac{1}{2}[\alpha(1-\alpha)(\beta_3(t)\alpha_2(t)-\beta_2(t)\alpha_3(t))],\quad p_2(t)=\tfrac{1}{2}[\alpha^2(\alpha_2(t)-\beta_2(t))+2\alpha\alpha_2(t)-\beta_3(t)-\alpha_3(t)]. \tag{35}$$

Introduce the notations

$$q_{11}=\frac{1}{2}-(\alpha_1^2+\beta_1^2+4\alpha_2^2+4\beta_2^2)-2(\alpha_1\alpha_2+\beta_1\beta_2)-16(\alpha_1\alpha_2+\beta_1\beta_2)^2,$$

$$q_{22}=\frac{1}{2}-(\alpha_3^2+\beta_3^2+4\alpha_2^2+4\beta_2^2)-2(\alpha_2\alpha_3+\beta_2\beta_3)-16(\alpha_2\alpha_2+\beta_2\beta_2)^2,$$

$$q_{12}=-(\alpha_2^2+\beta_2^2+2\beta_1\beta_3+2\alpha_1\alpha_3),$$

$$h_2=\max_{\overline{Q}}(|\alpha_{xx}|,|\beta_{xx}|,|\delta_{xx}|).$$

Theorem 2 *Assume that the condition* (34) *holds,*

$$\begin{gathered} c(t)\in C^1[0,T],\quad -c(t)\geq c_0\gg 0\quad \textit{for}\quad t\in[0,T], \\ h_i(x,t)\in C^3(\overline{Q}),\quad h_2<\tfrac{1}{2T},\quad \triangle(t)\neq 0, \end{gathered}$$

$$\begin{gathered} \left[\alpha_1(t)+\alpha_2(t)-4\alpha_2^2(t)\right]\xi_1^2+[-\beta_1(t)+\alpha_3(t)]\,\xi_1\xi_2-\left[\beta_3(t)+\beta_2(t)+4\beta_2^2(t)\right]\xi_2^2\geq 0, \\ q_{11}\xi_1^2+q_{12}\xi_1\xi_2+q_{22}\xi_2^2\geq 0\quad \textit{for}\quad t\in[0,T],\quad (\xi_1,\xi_2)\in\mathbb{R}^2, \end{gathered} \tag{36}$$

$$f(x,t)\in W_2^3(Q),\quad f_{xxttt}(x,t)\in L_2(Q).$$

Then there exists a regular solution to the problem (5), (2), (3), (6) *such that* $u(x,t)$, $u_{xx}(x,t)$ *belong to* $W_{2,x,t}^{2,3}(Q)$ *and* $q_k(t)\in L_2(0,T)$ $(k=1,2,3)$.

Proof. Consider the auxiliary boundary value problem: find a solution $u(x,t)$ to the equation

$$u_{ttt}+u_{xx}+c(t)u=\widehat{f}_{1xx}(x,t)+\lambda[\alpha_{xx}(x,t)u(0,t)+\beta_{xx}(x,t)u(\alpha,t)+\delta_{xx}(x,t)u(1,t)]$$

in Q such that

$$\begin{gathered} u_x(0,t)=\alpha_1(t)u(0,t)+\alpha_2(t)u(\alpha,t)+\alpha_3(t)u(1,t)+\alpha_0(t),\quad 0<t<T, \\ u_x(1,t)=\beta_1(t)u(0,t)+\beta_2(t)u(\alpha,t)+\beta_3(t)u(1,t)+\beta_0(t),\quad 0<t<T \end{gathered}$$

and

$$u(x,0)=u_t(x,0)=u(x,T)=0,\quad x\in\Omega.$$

As in Section 3, instead of (15)–(17) we consider the boundary value problem of finding a solution $v(x,t)$ to the equation

$$v_{ttt} + v_{xx} + c(t)v = f_2(x,t) + \lambda[\alpha_{xx}(x,t)v(0,t) + \beta_{xx}(x,t)v(\alpha,t) + \delta_{xx}(x,t)v(1,t)]$$

in Q such that

$$v_x(0,t) = \alpha_1(t)v(0,t) + \alpha_2(t)v(\alpha,t) + \alpha_3(t)v(1,t), \quad 0 < t < T,$$
$$v_x(1,t) = \beta_1(t)v(0,t) + \beta_2(t)v(\alpha,t) + \beta_3(t)v(1,t), \quad 0 < t < T,$$

$$v(x,0) = v_t(x,0) = v(x,T) = 0, \quad x \in \Omega, \tag{37}$$

where

$$f_2(x,t) = \widehat{f_1}_{xx}(x,t) + B_0(x,t), \quad B_0(x,t) = \lambda[\beta_{xx}(x,t)\gamma(\alpha,t) + \delta_{xx}(x,t)\gamma(1,t)] - \gamma_{ttt}(x,t) - \gamma_{xx}(x,t) - c(t)\gamma(x,t).$$

As before, without loss of generality we can consider the homogeneous initial conditions (37), assuming

$$\widehat{f_1}_x(0,0) = \widehat{f_1}_{xt}(0,0) = \widehat{f_1}_x(0,T) = 0, \quad \widehat{f_1}_x(1,0) = \widehat{f_1}_{xt}(1,0) = \widehat{f_1}_x(1,T) = 0.$$

For $(x,t) \in \overline{Q}$ and $\lambda \in [0,1]$ we put

$$\gamma_1(x,t,\lambda) = \tfrac{\lambda x^2}{2}[\beta_1(t) - \alpha_1(t)] + \lambda x\alpha_1(t), \quad \delta_1(x,t,\lambda) = \tfrac{\lambda x^2}{2}[\beta_2(t) - \alpha_2(t)] + \lambda x\alpha_2(t),$$
$$\epsilon_1(x,t,\lambda) = \tfrac{\lambda x^2}{2}[\beta_3(t) - \alpha_3(t)] + \lambda x\alpha_3(t),$$
$$w(x,t) = v(x,t) - \gamma_1(x,t,\lambda)v(0,t) - \delta_1(x,t,\lambda)v(\alpha,t) - \epsilon_1(x,t,\lambda)v(1,t),$$
$$v(x,t) = w(x,t) + \gamma_{11}(x,t,\lambda)w(0,t) + \delta_{11}(x,t,\lambda)w(\alpha,t) + \epsilon_{11}(x,t,\lambda)w(1,t),$$
$$\gamma_{11}(x,t,\lambda) = \gamma_1(x,t,\lambda)c_{11}(t) + \delta_1(x,t,\lambda)c_{21}(t) + \epsilon_1(x,t,\lambda)c_{31}(t),$$
$$\delta_{11}(x,t,\lambda) = \delta_1(x,t,\lambda)c_{22}(t) + \epsilon_1(x,t,\lambda)c_{32}(t), \quad \epsilon_{11}(x,t,\lambda) = \delta_1(x,t,\lambda)c_{23}(t) + \epsilon_1(x,t,\lambda)c_{33}(t),$$

where $c_{ij}(t)$ are the entries of the inverse matrix $C^{-1}(t)$ which is determined from the system of equations

$$C(t)\overrightarrow{v} = \overrightarrow{w}, \quad \overrightarrow{v} = (v(0,t), v(\alpha,t), v(1,t)), \quad \overrightarrow{w} = (w(0,t), w(\alpha,t), w(1,t)).$$

We have

$$C(t) = \begin{pmatrix} 1 & 0 & 0 \\ -\gamma_1(\alpha,t,\lambda) & 1 - \delta_1(\alpha,t,\lambda) & -\epsilon_1(\alpha,t,\lambda) \\ -\gamma_1(1,t,\lambda) & -\delta_1(1,t,\lambda) & 1 - \epsilon_1(1,t,\lambda) \end{pmatrix}, \quad \Delta(C) = \det|C(t)| = \lambda^2 p_1(t) + \lambda p_2(t) + 1,$$

where $p_1(t)$, $p_2(t)$ are given by (35). Notice that the determinant $\Delta(C) \neq 0$ when any of the three conditions (34) holds.

Let $v(x,t)$ be a solution to (18). In this case $w(x,t)$ satisfies the equation

$$w_{ttt} + w_{xx} + c(t)w = f_2(x,t) + \Phi(x,t,\lambda,\overline{w}(t)),$$

where

$$\overline{w}(t) = (\overrightarrow{w}_{ttt}, \overrightarrow{w}_{tt}, \overrightarrow{w}_t, \overrightarrow{w}) \equiv (w_1(t), w_2(t), \ldots, w_{12}(t)),$$
$$\overline{A}(x,t,\lambda) = (A_1(x,t,\lambda), A_2(x,t,\lambda), \ldots, A_{12}(x,t,\lambda)),$$
$$\Phi(x,t,\lambda,\overline{w}(t)) = (\overline{A}(x,t,\lambda), \overline{w}(t)) = \sum_{i=1}^{12} A_i(x,t,\lambda)w_i(t),$$
$$A_1(x,t,\lambda) = -\gamma_{11}(x,t,\lambda), \; A_2(x,t,\lambda) = -\delta_{11}(x,t,\lambda), \; A_3(x,t,\lambda) = -\epsilon_{11}(x,t,\lambda),$$
$$A_4(x,t,\lambda) = -3\gamma_{11\,t}(x,t,\lambda), \; A_5(x,t,\lambda) = -3\delta_{11\,t}(x,t,\lambda), \; A_6(x,t,\lambda) = -3\epsilon_{11\,t}(x,t,\lambda),$$
$$A_7(x,t,\lambda) = -3\epsilon_{11\,tt}(x,t,\lambda), \; A_8(x,t,\lambda) = -3\delta_{11\,tt}(x,t,\lambda), \; A_9(x,t,\lambda) = -3\epsilon_{11\,tt}(x,t,\lambda),$$
$$A_{10}(x,t,\lambda) = \lambda\alpha_{xx} - \gamma_{11\,ttt}(x,t,\lambda) - \gamma_{11\,xx}(x,t,\lambda) - c(t)\gamma_{11}(x,t,\lambda),$$
$$A_{11}(x,t,\lambda) = \lambda\beta_{xx} - \delta_{11\,ttt}(x,t,\lambda) - \delta_{11\,xx}(x,t,\lambda) - c(t)\delta_{11}(x,t,\lambda),$$
$$A_{12}(x,t,\lambda) = \lambda\delta_{xx} - \epsilon_{11\,ttt}(x,t,\lambda) - \epsilon_{11\,xx}(x,t,\lambda) - c(t)\epsilon_{11}(x,t,\lambda).$$

Consider the auxiliary boundary value problem: find a solution $w(x,t)$ to the equation

$$w_{ttt} + w_{xx} + c(t)w = f_2(x,t) + \Phi(x,t,\lambda,\overline{w}(t))$$

in Q such that

$$w_x(0,t) = w_x(1,t) = 0, \ \ t \in (0,T),$$
$$w(x,0) = w_t(x,0) = w(x,T) = 0, \ \ x \in \Omega.$$

As in Section 3, this problem is solvable in V_0. To justify this, we utilize the methods of regularization and continuation in a parameter. The function $v(x,t)$ thus constructed allows us to define a solution $u(x,t)$, $q_k(t)$ $(k = 1,2,3)$ to the boundary value problem (5), (2), (3), (6). □

CONCLUSIONS

1. Conditions (11) are some smallness conditions for Inverse Boundary Value Problem 1. Obviously, the set of data $f(x,t)$, $h_1(x,t)$, and $h_2(x,t)$ satisfying this condition is not empty. Similarly, the smallness conditions (34) for the Inverse Problem 2 also make some sense.

2. Note that the fulfillment of (13) implies the nonnegative definiteness of the quadratic form

$$\left[\tfrac{1}{2} - \alpha_1^2(t) - \beta_1^2(t)\right]\xi_1^2 - 2[\alpha_1(t)\alpha_2(t) + \beta_1(t)\beta_2(t)]\xi_1\xi_2 + \left[\tfrac{1}{2} - \alpha_2^2(t) - \beta_2^2(t)\right]\xi_2^2 \geq 0$$

for $t \in [0,T]$, $(\xi_1, \xi_2) \in \mathbb{R}^2$. Conditions (36) are also sufficient and they can be refined.

3. We can consider the inverse problem of finding together with a solution to the equation (1) the unknown coefficients $q_1(t)$, $q_2(t)$, $\dots$, $q_m(t)$ with pointwise overdetermination. In the study of the $(2s+1)$-th order equations in the time variable of the form (1) with the pointwise overdetermination the situation does not change, but the calculations will increase substantially.

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