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SINGULAR CAUCHY PROBLEM FOR GENERALIZED HOMOGENEOUS EULER—POISSON—DARBOUX EQUATION

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Abstract. In this paper, we solve singular Cauchy problem for a generalised form of an homogeneous Euler–Poisson–Darboux equation with constant potential, where Bessel operator acts instead of the each second derivative. In the classical formulation, the Cauchy problem for this equation is not correct. However, S. A. Tersenov observed that, considering the form of a general solution of the classical Euler–Poisson–Darboux equation, the derivative in the second initial condition must be multiplied by a power function whose degree is equal to the index of the Bessel operator acting on the time variable. The first initial condition remains in the usual formulation. With the chosen form of the initial conditions, the considering equation has a solution. Obtained solution is represented as the sum of two terms. The first tern is an integral containing the normalized Bessel function and the weighted spherical mean. The second term is expressed in terms of the derivative of the square of the time variable from the integral, which is similar in structure to the first term.

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1. Introduction

We study the initial value problem

$$Lu = \left[\sum_{i=1}^{n}\left(\frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}\right)-\left(\frac{\partial^2}{\partial t^2}+\frac{k}{t}\frac{\partial}{\partial t}\right)\right]u = c^2 u, \tag{1}$$

$$u(x,0;k)=\varphi(x), \quad \lim_{t\to+0} t^k u_t(x,t;k)=\psi(x), \quad u=u(x,t;k), \tag{2}$$

where $\gamma_i > 0$, $x_i > 0$, $i = 1,\dots,n$, $k \in \mathbb{R}$, $t > 0$. The equation (1) is called *generalized Euler–Poisson–Darboux equation* (*generalized EPD equation*).

Using the terminology from the book [1], a problem for the equation of the type

$$A(t)\frac{\partial^2 u}{\partial t^2}+B(t)\frac{\partial u}{\partial t}+C(t)u = Gu, \quad u=u(t,x), \quad x=(x_1,\dots,x_n),$$

where G is a linear operator, acting only by variables $x_1,\dots,x_n$, is called *singular* if at least one of the operator coefficients tends to infinity in some sense as $t \to 0$.

In [1] was given five general techniques for the solution of the singular Cauchy problem

$$\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}=\frac{\partial^2 u}{\partial t^2}+\frac{k}{t}\frac{\partial u}{\partial t}, \quad u=u(x,t;k), \tag{3}$$

$$u(x,0;k)=\varphi(x),\quad u_t(x,0;k)=0. \tag{4}$$

These methods are

(1) Fourier transforms method in distribution space,

(2) spectral technique in a Hilbert space,

(3) transmutation method,

(4) studying related simpler differential equations,

(5) energy methods.

Some of these methods were successfully applied to generalized EPD equation (1) and to $Lu=0$ in other pepers. Namely, using the Hankel transform instead of Fourier solutions to $Lu=0$ and (1) with conditions (4) were obtained in [2, 3], accordingly. The third and closely connected the forth methods were used to solution to $Lu=0$ in [2, 4]. In [5] transmutation method was used for obtaining new integral initial conditions for EPD equation (3). Abstract differential equations with Bessel operator of EPD type were studied in the [1], cf. also recent papers [6–8]. In [9] the problem (3), (2) was solved using 'descent' operators which are special cases of Buschman–Erdelyi transmutations operators (see [10, 11]). Here as a main result we obtain a solution to the problem (1), (2).

2. Basic definitions

In this section we need to introduce some of the basic definitions.

We deal with the subset of the Euclidean space

$$\mathbb{R}^{n+1}_+=\{(t,x)=(t,x_1,\ldots,x_n)\in\mathbb{R}^{n+1},\ t>0,\ x_1>0,\ldots,x_n>0\}.$$

Let $x=(x_1,\ldots,x_n)$, $|x|=\sqrt{\sum\limits_{i=1}^{n}x_i^2}$ and Ω be a finite or infinite open set in $\mathbb{R}^{n+1}$ symmetric with respect to each hyperplane $t=0$, $x_i=0$, $i=1,\ldots,n$, $\Omega_+=\Omega\cap\mathbb{R}^{n+1}_+$ and $\overline{\Omega}_+=\Omega\cap\overline{\mathbb{R}}^{n+1}_+$, where

$$\overline{\mathbb{R}}^{n+1}_+=\{(t,x)=(t,x_1,\ldots,x_n)\in\mathbb{R}^{n+1},\ t>0,\ x_1\geq 0,\ldots,x_n\geq 0\}.$$

Consider the class $C^m(\Omega_+)$ consisting of m times differentiable on Ω_+ functions and denote by $C^m(\overline{\Omega}_+)$ the subset of functions from $C^m(\Omega_+)$ such that all derivatives of these functions with respect to t and x_i for any $i=1,\ldots,n$ are continuous up to $t=0$ and $x_i=0$. Function $f\in C^m(\overline{\Omega}_+)$ we will call *even with respect to* t and x_i, $i=1,\ldots,n$ if

$$\left.\frac{\partial^{2k+1}f}{\partial t^{2k+1}}\right|_{t=0,x=0}=0,\quad \left.\frac{\partial^{2k+1}f}{\partial x_i^{2k+1}}\right|_{t=0,x=0}=0$$

for all nonnegative integer k (see [12, p. 21]. Class $C^m_{ev}(\overline{\Omega}_+)$ consists of functions from $C^m(\overline{\Omega}_+)$ even with respect to each variable t and x_i, $i=1,\ldots,n$. In the following we will denote $C^m_{ev}(\overline{\mathbb{R}}^{n+1}_+)$ by C^m_{ev}. We set

$$C^\infty_{ev}(\overline{\Omega}_+)=\bigcap C^m_{ev}(\overline{\Omega}_+)$$

with intersection taken for all finite m. Let $C^\infty_{ev}(\overline{\mathbb{R}}^{n+1}_+)=C^\infty_{ev}$.

We will deal with the *singular Bessel differential operator* B_ν (see, for example, [12, p. 5]:

$$(B_\nu)_t=\frac{\partial^2}{\partial t^2}+\frac{\nu}{t}\frac{\partial}{\partial t}=\frac{1}{t^\nu}\frac{\partial}{\partial t}t^\nu\frac{\partial}{\partial t},\quad t>0,$$

and the elliptical singular operator or the Laplace–Bessel operator Δ_γ:

$$\Delta_\gamma=(\Delta_\gamma)_x=\sum_{i=1}^n(B_{\gamma_i})_{x_i}=\sum_{i=1}^n\left(\frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x}\right)=\sum_{i=1}^n\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial}{\partial x_i}. \tag{5}$$

The operator (5) belongs to the class of *B-elliptic* operators by I. A. Kipriyanovs' classification (see [12, 13]). Operator

$$(\Box_{k,\gamma})_{t,x}=(B_k)_t-(\Delta_\gamma)_x$$

is a *B-hyperbolic* by the same classifications.

The symbol j_ν is used for the *normalized Bessel function*:

$$j_\nu(t)=\frac{2^\nu\Gamma(\nu+1)}{t^\nu}J_\nu(t),$$

where $J_\nu(t)$ is the Bessel function of the first kind of order ν (see [14]). The function $j_\nu(t)$ is even by t and

$$j_\nu(0)=1,\quad \frac{d}{dt}j_\nu(t)\bigg|_{t=0}=0. \tag{6}$$

Using formulas 9.1.27 from [15] we obtain

$$(B_\nu)_t j_{\frac{\nu-1}{2}}(\tau t)=-\tau^2 j_{\frac{\nu-1}{2}}(\tau t). \tag{7}$$

We deal with a multi-index $\gamma=(\gamma_1,\ldots,\gamma_n)$ which consists of positive fixed reals $\gamma_i>0$, $i=1,\ldots,n$, $|\gamma|=\gamma_1+\ldots+\gamma_n$.

The operator ${}^kT_t^\tau$ for $k>0$ is a *generalized translation* acts by a variable t defined by the next formula (see [16, p. 122, formula (5.19)])

$${}^kT_t^\tau f(t,x)=\frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{k}{2}\right)}\int\limits_0^\pi f(\sqrt{t^2+\tau^2-2t\tau\cos\varphi},x)\sin^{k-1}\varphi\,d\varphi \tag{8}$$

and ${}^\gamma\mathbf{T}^y={}^{\gamma_1}T^{y_1}_{x_1}\ldots{}^{\gamma_n}T^{y_n}_{x_n}$ is *multidimensional generalized translation*, where each of the one-dimensional generalized translations ${}^{\gamma_i}T^{y_i}_{x_i}$ acts by a variable x_i for $i=1,\ldots,n$ according to the formula (8).

It is known that

$${}^kT_t^\tau j_{\frac{k-1}{2}}(at)=j_{\frac{k-1}{2}}(at)j_{\frac{k-1}{2}}(a\tau), \tag{9}$$

where $a\in\mathbb{R}$.

Based on the multidimensional generalized translation ${}^\gamma T^y$ the *weighted spherical mean* $M_r^\gamma[f(x)]$ of a suitable function is constructed by the formula

$$M_r^\gamma[f(x)]=\frac{1}{|S_1^+(n)|_\gamma}\int\limits_{S_1^+(n)}{}^\gamma T_x^{r\theta}f(x)\theta^\gamma dS, \tag{10}$$

where

$$\theta^\gamma=\prod_{i=1}^n \theta_i^{\gamma_i},\quad S_1^+(n)=\{\theta:|\theta|=1,\theta{\in}\mathbb{R}_+^n\},\quad |S_1^+(n)|_\gamma=\frac{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}.$$

It is easy to see that

$$\lim_{r\to 0} M_r^\gamma[f(x)]=f(x),\quad \lim_{r\to 0}\frac{\partial}{\partial r}M_r^\gamma[f(x)]=0. \tag{11}$$

3. Singular initial value hyperbolic problems

We will be concerned with the solutions of the following singular initial value hyperbolic problem

$$Lu=\left[\sum_{i=1}^n\left(\frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}\right)-\left(\frac{\partial^2}{\partial t^2}+\frac{k}{t}\frac{\partial}{\partial t}\right)\right]u=c^2u, \tag{12}$$

$$u(x,0;k)=\varphi(x),\quad \lim_{t\to+0}t^k u_t(x,t;k)=\psi(x),\quad u=u(x,t;k).$$

We will call (12) the *generalized Euler–Poisson–Darboux equation.*

In [3] the next theorem was proven.

Theorem 1. *The solution $u\in C^2_{ev}(\mathbb{R}^{n+1}_+)$ to the*

$$[(\Delta_\gamma)_x-(B_k)_t]u=c^2u,\quad c>0,\quad u=u(x,t;k), \tag{13}$$

$$u(x,0;k)=\varphi(x),\quad u_t(x,0;k)=0, \tag{14}$$

for $k>n+|\gamma|-1$ is unique and defined by the formula

$$u(x,t;k)=A(n,\gamma,k)t^{1-k}\int\limits_0^t(t^2-r^2)^{\frac{k-n-|\gamma|-1}{2}}j_{\frac{k-n-|\gamma|-1}{2}}$$
$$\times(c\sqrt{t^2-r^2})r^{n+|\gamma|-1}M_r^\gamma[\varphi(x)]\,dr, \tag{15}$$

where

$$A(n,\gamma,k)=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)}.$$

It is known (see [1]) that if $u(x,t;k)$ is a solution to the (13) when the next two fundamental recursion formulas hold

$$u(x,t;k)=t^{1-k}u(x,t;2-k), \tag{16}$$

$$u(x,t;k)_t=tu(x,t;2+k). \tag{17}$$

Theorem 2. *Let* $\varphi=\varphi(x)$, $\varphi\in C_{ev}^{[\frac{n+|\gamma|-k}{2}]+2}$. *Then the solution of* (13), (14) *for* $k\leq n+|\gamma|-1$, $k\neq -1,-3,-5,\ldots$ *is*

$$u(x,t;k)=t^{1-k}\left(\frac{\partial}{t\partial t}\right)^m(t^{k+2m-1}u(x,t;k+2m)), \tag{18}$$

where m *is a minimum integer such that* $m\geq\frac{n+|\gamma|-k-1}{2}$ *and* $u(x,t;k+2m)$ *is the solution of the Cauchy problem*

$$[(\Delta_\gamma)_x-(B_{k+2m})_t]u=c^2u,\quad c>0, \tag{19}$$

$$u(x,0;k+2m)=\frac{\varphi(x)}{(k+1)(k+3)\ldots(k+2m-1)},\quad u_t(x,0;k+2m)=0. \tag{20}$$

Proof. In order to proof that (18) is a solution of (13), (14) when $k\leq n+|\gamma|-1$, $k\neq -1,-3,-5,\ldots$, we will use the recursion formulas (16) and (17). Let choose minimum integer m such that $k+2m>n+|\gamma|-1$. Now we can write the solution of the Cauchy problem

$$[(\Delta_\gamma)_x-(B_{k+2m})_t]u=c^2u,\quad c>0,$$

$$u(x,0;k+2m)=g(x),\quad u_t(x,0;k+2m)=0,\quad g\in C^2_{ev},$$

by (15). We have

$$u(x,t;k+2m)=A(n,\gamma,k+2m)t^{1-k-2m}\int_0^t(t^2-r^2)^{\frac{k+2m-n-|\gamma|-1}{2}}$$
$$\times j_{\frac{k+2m-n-|\gamma|-1}{2}}(c\sqrt{t^2-r^2})r^{n+|\gamma|-1}M_r^\gamma[g(x)]\,dr,$$

where

$$A(n,\gamma,k+2m)=\frac{2\Gamma\left(\frac{k+2m+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{k+2m-n-|\gamma|+1}{2}\right)}.$$

Considering (16), it is easy to see that

$$t^{k+2m-1}u(x,t;k+2m)=u(x,t;2-k-2m).$$

Applying (17) to the last formula m times we get

$$\left(\frac{\partial}{t\partial t}\right)^m(t^{k+2m-1}u(x,t;k+2m))=u(x,t;2-k).$$

Applying again (16) we can write

$$u(x,t;k)=t^{1-k}\left(\frac{\partial}{t\partial t}\right)^m(t^{k+2m-1}u(x,t;k+2m)), \tag{21}$$

which gives the solution of the (19). Now we obtain the function g such that the (20) is true. From (21) it follows that

$$u(x,t;k)=(k+1)(k+3)\ldots(k+2m-1)u(x,t;k+2m)+Ctu(x,t;k+2m)+O(t^2),$$

when $t\to 0$, where C is a constant. Evidently, if

$$g(x)=\frac{\varphi(x)}{(k+1)(k+3)\dots(k+2m-1)}$$

then $u(x,t;k)$ defined by (18) satisfies the initial conditions (14).

Let us recall that for $u(x,t;k+2m)$ to be a solution of (19), (20) it is sufficient that $f\in C^2_{ev}$. In order to be able to carry out the construction (21), it is sufficient to require that $f\in C_{ev}^{[\frac{n+|\gamma|-k}{2}]+2}$.

Theorem 3. *Let* $\psi\in C_{ev}^{[\frac{n+|\gamma|+k-1}{2}]}$. *The solution* $u=u(x,t;k)$ *to the*

$$[(\Delta_\gamma)_x-(B_k)_t]u=c^2u,\quad c>0, \tag{22}$$

$$u(x,0;k)=0,\quad \lim_{t\to+0}t^k u_t(x,t;k)=\psi(x), \tag{23}$$

for $k<1$ *is defined by the formula*

$$u(t,x;k)=B(n,\gamma,k,q)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q\left(\int\limits_0^t (t^2-r^2)^{\frac{1-k+2q-n-|\gamma|}{2}}\right.$$

$$\left.\times\, j_{\frac{1-k+2q-n-|\gamma|}{2}}(c\sqrt{t^2-r^2})r^{n+|\gamma|-1}M_r^\gamma[\psi(x)]\,dr\right), \tag{24}$$

where

$$B(n,\gamma,k,q)=\frac{2^{-q}\Gamma\left(\frac{1-k}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{2-k+2q-n-|\gamma|+1}{2}\right)}.$$

Proof. Let $q\geq 0$ is the smallest positive integer number such that $2-k+2q>n+|\gamma|-1$ i.e. $q>\frac{n+|\gamma|+k-3}{2}$ and let $u(x,t;2-k+2q)$ is a solution to (22) when we take $2-k+2q$ instead of k such that

$$u(x,0;2-k+2q)=\psi(x),\quad u_t(x,0;2-k+2q)=0. \tag{25}$$

When by property (16) we obtain

$$u(t,x;k-2q)=t^{1-k+2q}u(t,x;2-k+2q)$$

is a solution to the equation

$$(\Delta_\gamma)_x u-\frac{\partial^2 v}{\partial t^2}-\frac{k-2q}{t}\frac{\partial v}{\partial t}=c^2u.$$

Further, applying q-times the formula (17) we obtain

$$\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q u(t,x;k-2q)=\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q\left(t^{1-k+2q}u(t,x;2-k+2q)\right)$$

is a solution to the (23). In order to get a solution to (22) satisfying to the conditions (23) we use the multiplier $\frac{2^{-q}\Gamma(\frac{3-k}{2})}{(1-k)\Gamma(\frac{3-k+2q}{2})}$. Let

$$u(t,x;k)=\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q(t^{1-k+2q}u(t,x;2-k+2q)). \tag{26}$$

We have shown that (26) satisfies the equation (22). Now we will prove that $u(t,x;k)$ satisfies the conditions (25). Using formula 1.13 from [17, p. 9] we obtain

$$\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q (t^{1-k+2q}u(t,x;2-k+2q)) = \sum_{s=0}^{q} \frac{2^{q-s}C_q^s\Gamma\left(\frac{1-k}{2}+q+1\right)}{\Gamma\left(\frac{1-k}{2}+s+1\right)} t^{1-k+2s}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s u(t,x;2-k+2q),$$

and $u(0,x;k)=0$ for $k<1$. For the second condition (25) we get

$$\begin{aligned}
&\lim_{t\to 0} t^k u_t(t,x;k) \\
&= \frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)} \lim_{t\to 0} t^k \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q (t^{1-k+2q}u(t,x;2-k+2q)) \\
&= \frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)} \lim_{t\to 0} t^k \frac{\partial}{\partial t}\sum_{s=0}^{q} \frac{2^{q-s}C_q^s\Gamma\left(\frac{1-k}{2}+q+1\right)}{\Gamma\left(\frac{1-k}{2}+s+1\right)} t^{1-k+2s} \\
&\quad \times \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s u(t,x;2-k+2q) \\
&= \frac{1}{1-k}\lim_{t\to 0} t^k \frac{\partial}{\partial t}\left(t^{1-k}u(t,x;2-k+2q)\right) \\
&= \frac{1}{1-k}\lim_{t\to 0} t^k \left((1-k)t^{-k}u(t,x;2-k+2q)+t^{1-k}u_t(t,x;2-k+2q)\right) \\
&= \frac{1}{1-k}\lim_{t\to 0} ((1-k)u(t,x;2-k+2q)+tu_t(t,x;2-k+2q)) \\
&= \lim_{t\to 0} u(t,x;2-k+2q) = \psi(x).
\end{aligned}$$

Now we write the representation of $u(t,x;k)$ through the integral. Using formula (15) we get

$$u(x,t;2-k+2q) = A(n,\gamma,2-k+2q)t^{k-1-2q}\int_0^t (t^2-r^2)^{\frac{1-k+2q-n-|\gamma|}{2}} \times j_{\frac{1-k+2q-n-|\gamma|}{2}}(c\sqrt{t^2-r^2})r^{n+|\gamma|-1}M_r^\gamma[\psi(x)]\,dr.$$

Considering (26) we write

$$u(t,x;k) = A(n,\gamma,2-k+2q)\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)} \times \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q \int_0^t (t^2-r^2)^{\frac{1-k+2q-n-|\gamma|}{2}} j_{\frac{1-k+2q-n-|\gamma|}{2}}(c\sqrt{t^2-r^2})r^{n+|\gamma|-1}M_r^\gamma[\psi(x)]\,dr.$$

Simplifying we get (24) and this completes the proof.

The union of the theorems 2 and 3 gives the following statement.

Theorem 4. *Let* $\varphi=\varphi(x)$, $\varphi\in C_{ev}^{[\frac{n+|\gamma|-k}{2}]+2}$, $\psi=\psi(x)$, $\psi\in C_{ev}^{[\frac{n+|\gamma|+k-1}{2}]}$. *Then the solution of*

$$[(\Delta_\gamma)_x-(B_k)_t]u=c^2u,\quad c>0, \tag{27}$$

$$u(x,0;k)=\varphi(x),\quad \lim_{t\to+0}t^ku_t(x,t;k)=\psi(x), \tag{28}$$

for $k\le\min\{n+|\gamma|-1,1\}$, $k\ne-1,-3,-5,\dots$ *is given by formula*

$$u(x,t;k)=u_1(x,t;k)+u_2(x,t;k),$$

where $u_1(x,t;k)$ *is found by Theorem 2 and* $u_1(x,t;k)$ *is found by Theorem 3.*

4. Examples

1. Let's start with an example

$$[(B_\gamma)_x-(B_k)_t]u=c^2u,\quad c>0,\quad u=u(t,x;k),\quad (t,x)\in\mathbb{R}^2_+, \tag{29}$$

$$u(x,0;k)=j_\gamma(ax),\quad u_t(x,0;k)=0, \tag{30}$$

where $n=1$, $\gamma-2\le k\le\gamma$, $k\ne-1$, $0<\gamma$, $a\in\mathbb{R}$. When $m=1$ and considering that (see (9))

$${}^\gamma T^r_x j_\gamma(ax)=j_{\frac{\gamma-1}{2}}(ax)j_{\frac{\gamma-1}{2}}(ar) \tag{31}$$

we obtain

$$u(x,t;k+2)=\frac{A(1,\gamma,k+2)}{k+1}j_{\frac{\gamma-1}{2}}(ax)t^{-k-1}\times\int\limits_0^t(t^2-r^2)^{\frac{k-\gamma}{2}}j_{\frac{k-\gamma}{2}}(c\sqrt{t^2-r^2})r^\gamma j_{\frac{\gamma-1}{2}}(ar)\,dr,$$

where

$$A(1,\gamma,k+2)=\frac{2\Gamma\left(\frac{k+3}{2}\right)}{\Gamma\left(\frac{1+\gamma}{2}\right)\Gamma\left(\frac{k+2-\gamma}{2}\right)}.$$

Passing to the functions J_ν in the integral we get

$$u(x,t;k+2)=\frac{2^{\frac{k-1}{2}}A(1,\gamma,k+2)\Gamma\left(\frac{1+\gamma}{2}\right)\Gamma\left(1+\frac{k-\gamma}{2}\right)}{(k+1)a^{\frac{\gamma-1}{2}}c^{\frac{k-\gamma}{2}}}j_{\frac{\gamma-1}{2}}(ax)t^{-k-1}\times\int\limits_0^t(t^2-r^2)^{\frac{k-\gamma}{4}}J_{\frac{k-\gamma}{2}}(c\sqrt{t^2-r^2})r^{\frac{\gamma-1}{2}+1}J_{\frac{\gamma-1}{2}}(ar)\,dr.$$

Applying the formula 2.12.35.2 from [18] of the form

$$\int\limits_0^t(t^2-x^2)^{m+\frac{\mu}{2}}x^{\nu+1+2l}J_\mu(c\sqrt{t^2-x^2})J_\nu(hx)dx=t^{\mu+\nu-m-l+1}c^\mu h^\nu\times\left(\frac{\partial}{c\partial c}\right)^m\left(\frac{\partial}{h\partial h}\right)^l[(c^2+h^2)^{-\frac{\mu+\nu+m+l+1}{2}}J_{\mu+\nu+m+l+1}(t\sqrt{c^2+h^2})],$$

$$t>0,\quad \operatorname{Re}\nu>-l-1,\quad \operatorname{Re}\mu>-m-1, \tag{32}$$

we obtain

$$\int_0^t (t^2-r^2)^{\frac{k-\gamma}{4}} J_{\frac{k-\gamma}{2}}(c\sqrt{t^2-r^2})r^{\frac{\gamma-1}{2}+1}J_{\frac{\gamma-1}{2}}(ar)\,dr$$
$$= t^{\frac{k+1}{2}}a^{\frac{\gamma-1}{2}}c^{\frac{k-\gamma}{2}}(\sqrt{a^2+c^2})^{-\frac{k+1}{2}}J_{\frac{k+1}{2}}(t\sqrt{a^2+c^2})$$

and

$$u(x,t;k+2) = \frac{2^{\frac{k+1}{2}}\Gamma\left(\frac{k+3}{2}\right)}{k+1} j_{\frac{\gamma-1}{2}}(ax)t^{-\frac{k+1}{2}}(\sqrt{a^2+c^2})^{-\frac{k+1}{2}}J_{\frac{k+1}{2}}(t\sqrt{a^2+c^2})$$
$$= \frac{1}{k+1} j_{\frac{\gamma-1}{2}}(ax) j_{\frac{k+1}{2}}(t\sqrt{a^2+c^2}).$$

Then the solution of (29), (30) is

$$u(x,t;k) = t^{1-k}\frac{\partial}{t\partial t}(t^{k+1}u(x,t;k+2)) = \frac{t^{1-k}}{k+1} j_{\frac{\gamma-1}{2}}(ax)\frac{\partial}{t\partial t}(t^{k+1}j_{\frac{k+1}{2}}(t\sqrt{a^2+c^2}))$$
$$= \frac{2^{\frac{k+1}{2}}t^{1-k}\Gamma\left(\frac{k+3}{2}\right)}{(k+1)(\sqrt{a^2+c^2})^{\frac{k+1}{2}}} j_{\frac{\gamma-1}{2}}(ax)\frac{\partial}{t\partial t}(t^{\frac{k+1}{2}}J_{\frac{k+1}{2}}(t\sqrt{a^2+c^2}))$$
$$= \frac{2^{\frac{k-1}{2}}\Gamma\left(\frac{k+1}{2}\right)}{(\sqrt{a^2+c^2})^{\frac{k-1}{2}}} j_{\frac{\gamma-1}{2}}(ax)t^{\frac{1-k}{2}}J_{\frac{k-1}{2}}(t\sqrt{a^2+c^2}) = j_{\frac{\gamma-1}{2}}(ax)j_{\frac{k-1}{2}}(t\sqrt{a^2+c^2}).$$

As might be seen from (7) and (6)

$$j_{\frac{\gamma-1}{2}}(ax)\lim_{t\to 0} j_{\frac{k-1}{2}}(t\sqrt{a^2+c^2}) = j_{\frac{\gamma-1}{2}}(ax),$$
$$j_{\frac{\gamma-1}{2}}(ax)\lim_{t\to 0}\frac{\partial}{\partial t} j_{\frac{k-1}{2}}(t\sqrt{a^2+c^2}) = j_{\frac{\gamma-1}{2}}(ax) = 0,$$
$$(B_\gamma)_x j_{\frac{\gamma-1}{2}}(ax)j_{\frac{k-1}{2}}(t\sqrt{a^2+c^2}) = -a^2 j_{\frac{\gamma-1}{2}}(ax)j_{\frac{k-1}{2}}(t\sqrt{a^2+c^2}),$$
$$(B_k)_t j_{\frac{\gamma-1}{2}}(ax)j_{\frac{k-1}{2}}(t\sqrt{a^2+c^2}) = -(a^2+c^2) j_{\frac{\gamma-1}{2}}(ax)j_{\frac{k-1}{2}}(t\sqrt{a^2+c^2})$$

which showss that the function

$$u(t,x;k) = j_{\frac{\gamma-1}{2}}(ax)j_{\frac{k-1}{2}}(t\sqrt{a^2+c^2}) \tag{33}$$

satisfies (29), (30).

2. Consider the problem

$$[(B_\gamma)_x - (B_k)_t]u = c^2u, \quad c>0, \quad u=u(t,x;k), \quad (t,x)\in\mathbb{R}^2_+, \tag{34}$$
$$u(x,0;k)=0, \quad \lim_{t\to+0} t^k u_t(x,t;k) = j_\gamma(bx), \tag{35}$$

where $n=1$, $k<1$, $0<\gamma<3$, $b\in\mathbb{R}$. When $q=1$ and considering (31) we obtain

$$u(t,x;k) = B(1,\gamma,k,1)j_{\frac{\gamma-1}{2}}(bx)$$
$$\times\frac{1}{t}\frac{\partial}{\partial t}\left(\int_0^t (t^2-r^2)^{1-\frac{k+\gamma}{2}} j_{1-\frac{k+\gamma}{2}}(c\sqrt{t^2-r^2})j_{\frac{\gamma-1}{2}}(br)r^\gamma\,dr\right)$$

$$= \frac{B(1,\gamma,k,1)\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(2-\frac{k+\gamma}{2}\right)}{2^{\frac{k-1}{2}}b^{\frac{\gamma-1}{2}}c^{1-\frac{k+\gamma}{2}}} j_{\frac{\gamma-1}{2}}(bx)\frac{1}{t}\frac{\partial}{\partial t}\left(\int\limits_0^t (t^2-r^2)^{\frac{2-k-\gamma}{4}} \times J_{1-\frac{k+\gamma}{2}}(c\sqrt{t^2-r^2})J_{\frac{\gamma-1}{2}}(br)r^{\frac{\gamma-1}{2}+1}\,dr\right),$$

where

$$B(1,\gamma,k,1) = \frac{\Gamma\left(\frac{1-k}{2}\right)}{2\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(2-\frac{k+\gamma}{2}\right)}.$$

Applying the formula 2.12.35.2 from [18] of the form

$$\int\limits_0^t (t^2-x^2)^{m+\frac{\mu}{2}}x^{\nu+1+2l}J_\mu(c\sqrt{t^2-x^2})J_\nu(hx)\,dx$$

$$= t^{\mu+\nu-m-l+1}c^\mu h^\nu\left(\frac{\partial}{c\partial c}\right)^m\left(\frac{\partial}{h\partial h}\right)^l[(c^2+h^2)^{-\frac{\mu+\nu+m+l+1}{2}} \times J_{\mu+\nu+m+l+1}(t\sqrt{c^2+h^2})],$$

$$t>0,\quad \mathrm{Re}\,\nu > -l-1,\quad \mathrm{Re}\,\mu > -m-1,$$

we obtain

$$\int\limits_0^t (t^2-r^2)^{\frac{2-k-\gamma}{4}}J_{1-\frac{k+\gamma}{2}}(c\sqrt{t^2-r^2})J_{\frac{\gamma-1}{2}}(br)r^{\frac{\gamma-1}{2}+1}\,dr$$

$$= b^{\frac{\gamma-1}{2}}c^{1-\frac{k+\gamma}{2}}(b^2+c^2)^{\frac{k-3}{4}}t^{\frac{3-k}{2}}J_{\frac{3-k}{2}}(t\sqrt{b^2+c^2})$$

and

$$u(t,x;k) = 2^{\frac{1-k}{2}}B(1,\gamma,k,1)\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(2-\frac{k+\gamma}{2}\right)(b^2+c^2)^{\frac{k-3}{4}}j_{\frac{\gamma-1}{2}}(bx) \times \frac{1}{t}\frac{\partial}{\partial t}(t^{\frac{3-k}{2}}J_{\frac{3-k}{2}}(t\sqrt{b^2+c^2}))$$

$$= 2^{-\frac{k+1}{2}}\Gamma\left(\frac{1-k}{2}\right)(\sqrt{b^2+c^2})^{\frac{k-1}{2}}j_{\frac{\gamma-1}{2}}(bx)t^{\frac{1-k}{2}}J_{\frac{1-k}{2}}(t\sqrt{b^2+c^2})$$

$$= \frac{t^{1-k}}{1-k}j_{\frac{\gamma-1}{2}}(bx)j_{\frac{1-k}{2}}(t\sqrt{b^2+c^2}).$$

Taking into account (7) and (6) it is easy to check that

$$(B_\gamma)_x\frac{t^{1-k}}{1-k}j_{\frac{\gamma-1}{2}}(bx)j_{\frac{1-k}{2}}(t\sqrt{b^2+c^2}) = -b^2\frac{t^{1-k}}{1-k}j_{\frac{\gamma-1}{2}}(bx)j_{\frac{1-k}{2}}(t\sqrt{b^2+c^2}),$$

$$(B_k)_t\frac{t^{1-k}}{1-k}j_{\frac{\gamma-1}{2}}(bx)j_{\frac{1-k}{2}}(t\sqrt{b^2+c^2}) = -(b^2+c^2)\frac{t^{1-k}}{1-k}j_{\frac{\gamma-1}{2}}(bx)j_{\frac{1-k}{2}}(t\sqrt{b^2+c^2}),$$

$$\lim_{t\to 0}\frac{t^{1-k}}{1-k}j_{\frac{\gamma-1}{2}}(bx)j_{\frac{1-k}{2}}(t\sqrt{b^2+c^2}) = 0$$

and

$$j_{\frac{\gamma-1}{2}}(bx)\lim_{t\to 0} t^k \frac{\partial}{\partial t}\left(\frac{t^{1-k}}{1-k} j_{\frac{1-k}{2}}(t\sqrt{b^2+c^2})\right) = j_{\frac{\gamma-1}{2}}(bx)$$

which confirms that the function

$$u(t,x;k) = \frac{t^{1-k}}{1-k} j_{\frac{\gamma-1}{2}}(bx) j_{\frac{1-k}{2}}(t\sqrt{b^2+c^2}) \tag{36}$$

satisfies (34), (35).

3. From examples 1 and 2 it is plain to see that the solutions of

$$[(B_\gamma)_x - (B_k)_t]u = c^2 u, \quad c>0, \quad u=u(t,x;k), \quad (t,x)\in \mathbb{R}^2_+, \tag{37}$$

$$u(x,0;k) = j_\gamma(ax), \quad \lim_{t\to+0} t^k u_t(x,t;k) = j_\gamma(bx), \tag{38}$$

where $n=1$, $0<\gamma<1$ $\gamma-2\le k\le\gamma$, $k\neq -1$, $a,b\in\mathbb{R}$ is

$$u(t,x;k) = j_{\frac{\gamma-1}{2}}(ax) j_{\frac{k-1}{2}}(t\sqrt{a^2+c^2}) + \frac{t^{1-k}}{1-k} j_{\frac{\gamma-1}{2}}(bx) j_{\frac{1-k}{2}}(t\sqrt{b^2+c^2}). \tag{39}$$

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