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WEYL–ALMOST PERIODIC AND ASYMPTOTICALLY WEYL–ALMOST PERIODIC PROPERTIES OF SOLUTIONS TO LINEAR AND SEMILINEAR ABSTRACT VOLTERRA INTEGRO–DIFFERENTIAL EQUATIONS

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Abstract The main purpose of paper is to consider Weyl-almost periodic and asymptotically Weyl-almost periodic solutions of linear and semilinear abstract Volterra integro-differential equations. We focus our attention to the investigations of Weyl-almost periodic and asymptotically Weyl-almost periodic properties of both, finite and infinite convolution product, working in the setting of complex Banach spaces. We introduce the class of asymptotically (equi)-Weyl-p-almost periodic functions depending on two parametres and prove a composition principle for the class of asymptotically equi-Weyl-p-almost periodic functions. Basically, our results are applicable in any situations where the variation of parameters formula takes a role. We provide several new contributions to abstract linear and semilinear Cauchy problems, including equations with the Weyl-Liouville fractional derivatives and the Caputo fractional derivatives. We provide some applications of our abstract theoretical results at the end of paper, considering primarily abstract degenerate differential equations, including the famous Poisson heat equation and its fractional analogues.

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1. Introduction and Preliminaries

It is well known that the notion of an almost periodic function was introduced by H. Bohr around 1924–1926 and later generalized by V. V. Stepanov, H. Weyl, A. S. Besicovitch and many other mathematicians. In this paper, we primarily consider Weyl's and Kovanko's classes of generalized almost periodic functions, which were introduced in 1927 and 1944, respectively. The analysis of various classes of generalized almost periodic solutions of abstract Volterra integro-differential equations is still very attractive field of research of many mathematicians. For more details on the subject, we refer the reader to the monographs [1] by D. Cheban, [2] by T. Diagana, [3] by G. M. N'Guérékata and [4] by the author.

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The main purpose of this paper is to analyze the existence and uniqueness of Weyl-almost periodic solutions and asymptotically Weyl-almost periodic solutions of abstract linear and semilinear Volterra integro-differential equations. In such a way, we continue the recent research studies [5] by F. Bedouhene, Y. Ibaouene, O. Mellah, R. de Fitte and [6] by the author.

The organization of paper is briefly described as follows. The introductory section contains three separate subsections, devoted to the study of degenerate semigroups and degenerate fractional resolvent families generated by multivalued linear operators, Stepanov almost periodic functions and asymptotically Stepanov almost periodic functions, Weyl almost periodic functions and asymptotically Weyl almost periodic functions. Our main results are given in Section 2 and Section 3, while Section 4 is deserved for illustrative examples and applications of our results.

Concerning the ideas and novelties of this paper, the following should be said. In our forthcoming monograph [4], we have analyzed generalized almost periodic and generalized asymptotically almost periodic solutions of various classes of abstract Volterra integro-differential equations, degenerate or non-degenerate in time variable. The solution operator families examined in [4] are generally degenerate in time and have removable singularities at zero (see [7–15]) for the theory of abstract degenerate Volterra integro-differential equations in Banach and locally convex spaces). Especially, in [16], the Weyl-p-almost periodic and asymptotically Weyl-p-almost periodic properties of convolution products have been analyzed in the case that $p = 1$. Motivated by recent research results obtained in [5], where the case $p > 1$ has been considered for infinite convolution product, and strongly continuous exponentially decaying semigroups, we expand our results from [6] to the general case $p \geq 1$, and for solution operator families which may not be strongly continuous at zero (Proposition 2.1). In contrast to [5], we also consider the finite convolution product in Proposition 2.5. The class of asymptotically Weyl-p-almost periodic functions has been recently introduced in [6]. As mentioned in the abstract, in this paper, we introduce the class of asymptotically (equi)-Weyl-p-almost periodic functions depending on two parametres and prove a composition principle for the class of asymptotically equi-Weyl-p-almost periodic functions (Definition 3.3, Theorem 3.4). Although our main contributions are given for linear problems, in Theorem 3.5 and Theorem 3.6, we analyze certain classes of abstract semilinear Cauchy problems having unique solutions in the class of bounded continuous functions. The question when the obtained solutions will be (asymptotically) equi-Weyl-p-almost periodic for a given equi-Weyl-p-almost periodic forcing term $f(\cdot,\cdot)$ is non-trivial and we will consider this question for degenerate resolvent operator families with certain growth rates at zero and plus infinity somewhere else.

Before explaining the notation used in the paper, the author would like to express his sincere thanks to Prof. V. Fedorov (Chelyabinsk, Russia) for many stimulating discussions during the research.

By X and Y we denote two non-trivial Banach spaces over the field of complex numbers. The symbol $L(X, Y)$ denotes the space consisting of all continuous linear mappings from X into Y; $L(X) \equiv L(X,X)$. Let $I = \mathbb{R}$ or $I = [0,\infty)$. By $C_b(I : X)$ and $BUC(I : X)$ we denote the vector spaces consisting of all bounded continuous functions from I into X and all bounded continuous functions from I into X vanishing at infinity, respectively. Endowed with the usual sup-norm, any of these spaces becomes one of Banach's.

Fractional calculus and fractional differential equations are rapidly growing areas of research, founding applications in diverse fields of theoretical and applied science (see e.g. [17–21] for the basic theory). The Mittag-Leffler functions and Wright functions are, without any doubt, the most important special functions in fractional calculus. Let $\gamma \in (0,1)$. The Wright function $\Phi_\gamma(\cdot)$ is defined by the formula

$$\Phi_\gamma(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1-\gamma-\gamma n)}, \quad z \in \mathbb{C}.$$

We know that $\Phi_\gamma(\cdot)$ is an entire function, as well as that $\Phi_\gamma(t) \geq 0$, $t \geq 0$. For further information concerning the Mittag-Leffler and Wright functions, one may refer e.g. to the doctoral dissertation of E. Bazhlekova [17] and references cited therein.

Let $s \in \mathbb{R}$. Define $\lfloor s \rfloor := \sup\{l \in \mathbb{Z} : s \geq l\}$ and $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. If $\alpha > 0$, then we define

$$g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0.$$

In this paper, we use the Weyl-Liouville fractional derivatives $D_{t,+}^{\gamma}u(t)$ of order $\gamma \in (0,1)$ and the Caputo fractional derivatives of order $\alpha > 0$. The Weyl–Liouville fractional derivatives $D_{t,+}^{\gamma}u(t)$ of order $\gamma \in (0,1)$ is defined for those continuous functions $u : \mathbb{R} \to X$ such that

$$t \mapsto \int_{-\infty}^{t} g_{1-\gamma}(t-s)u(s)\,ds, \quad t \in \mathbb{R},$$

is a well-defined continuously differentiable mapping, by

$$D_{t,+}^{\gamma}u(t) := \frac{d}{dt}\int_{-\infty}^{t} g_{1-\gamma}(t-s)u(s)\,ds, \quad t \in \mathbb{R}.$$

For more details about this type of fractional derivatives, we refer the reader to the paper [22] by J. Mu, Y. Zhoa and L. Peng. If $\alpha > 0$ and $m = \lceil \alpha \rceil$, the Caputo fractional derivative $\mathbf{D}_t^{\alpha}u(t)$ is defined for those functions $u \in C^{m-1}([0,\infty) : X)$ for which

$$g_{m-\alpha} * \left(u - \sum_{k=0}^{m-1} u_k g_{k+1}\right) \in C^m([0,\infty) : X),$$

by

$$\mathbf{D}_t^{\alpha}u(t) = \frac{d^m}{dt^m}\left[g_{m-\alpha} * \left(u - \sum_{k=0}^{m-1} u_k g_{k+1}\right)\right].$$

Fractional differential equations with Caputo derivatives have been investigated, among many other research papers and monographs, in [13, 18, 19].

1.1. Degenerate semigroups and degenerate fractional resolvent families generated by multivalued linear operators. Multivalued linear operators in Banach spaces have been analyzed by many authors. A multivalued linear operator (MLO, for short) $\mathscr{A} : X \to P(Y)$ is any mapping satisfying the following conditions:

(i) $D(\mathscr{A}) := \{x \in X : \mathscr{A}x \neq \varnothing\}$ is a linear subspace of X;

(ii) $\mathscr{A}x + \mathscr{A}y \subseteq \mathscr{A}(x+y)$, $x, y \in D(\mathscr{A})$ and $\lambda\mathscr{A}x \subseteq \mathscr{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathscr{A})$.

If $X = Y$, then we say that $\mathscr{A}$ is an MLO in X.

It is well known that, if $x, y \in D(\mathscr{A})$ and $\lambda, \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, then $\lambda\mathscr{A}x + \eta\mathscr{A}y = \mathscr{A}(\lambda x + \eta y)$. IF $\mathscr{A}$ is an MLO, then $\mathscr{A}0$ is a linear submanifold of Y and $\mathscr{A}x = f + \mathscr{A}0$ for any $x \in D(\mathscr{A})$ and $f \in \mathscr{A}x$. Put

$$R(\mathscr{A}) := \{\mathscr{A}x : x \in D(\mathscr{A})\}.$$

The *resolvent set* of $\mathscr{A}$, $\rho(\mathscr{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(\lambda - \mathscr{A}) = X$;

(ii) $R(\lambda : \mathscr{A}) \equiv (\lambda - \mathscr{A})^{-1}$ is a single-valued bounded operator on X.

It is well known that $\rho(\mathscr{A})$ is an open subset of $\mathbb{C}$. The operator $\lambda \mapsto R(\lambda : \mathscr{A})$ is called the resolvent of $\mathscr{A}$ ($\lambda \in \rho(\mathscr{A})$). For further information regarding multivalued linear operators, we refer the reader to the monographs [23] by R. Cross and [10] by A. Favini, A. Yagi.

The basic source of information about degenerate (a, k)-regularized C-resolvent families may be obtained by consulting the forthcoming monograph [13]. Suppose that a closed MLO $\mathscr{A}$ satisfies the condition [10, p. 47] introduced by A. Favini and A. Yagi:

(P) There exist finite constants $c, M > 0$ and $\beta \in (0, 1]$ such that

$$\Psi := \Psi_c := \Big\{\lambda \in \mathbb{C} : \Re\lambda \geq -c(|\Im\lambda| + 1)\Big\} \subseteq \rho(\mathscr{A})$$

and

$$\|R(\lambda : \mathscr{A})\| \leq M(1 + |\lambda|)^{-\beta}, \quad \lambda \in \Psi.$$

If this condition holds and $\beta > \theta$, then degenerate strongly continuous semigroup $(T(t))_{t>0} \subseteq L(X)$ generated by $\mathscr{A}$ satisfies estimate

$$\|T(t)\| \leq M_0 e^{-ct} t^{\beta-1}, \quad t > 0,$$

for some finite constant $M_0 > 0$ [24].

For our considerations of the abstract degenerate differential equations with Weyl–Liouville or Caputo fractional derivatives, we need some preliminaries concerning subordination fractional resolvent operator families. Assume that $\gamma \in (0, 1]$. Set

$$T_{\gamma,\nu}(t)x := t^{\gamma\nu} \int_0^\infty s^\nu \Phi_\gamma(s) T(st^\gamma)x\, ds, \quad t > 0,\ x \in X,$$

$$S_\gamma(t) := T_{\gamma,0}(t),\ t > 0; \quad S_\gamma(0) := I,$$

and

$$P_\gamma(t) := \frac{\gamma T_{\gamma,1}(t)}{t^\gamma}, \quad t > 0.$$

It is well known that, if $x_0 \in X$ is a point of continuity of $(T(t))_{t>0}$, then x_0 is a point of continuity of subordintated fractional resolvent family $(S_\gamma(t))_{t>0}$, and due to the estimate

$$\|S_\gamma(t)\| \leq M_1 t^{\gamma(\beta-1)}, \quad t > 0,$$

for some finite constant $M_1 > 0$, the mapping

$$t \mapsto S_\gamma(t)x_0, \quad t \geq 0,$$

is continuous and tending to zero as t tends to $+\infty$. Furthermore, we have:

$$\|R_\gamma(t)\| \leq M_1 t^{\gamma\beta-1},\ t \in (0,1] \ \text{ and } \ \|R_\gamma(t)\| \leq M_2 t^{-1-\gamma},\ t \geq 1.$$

Of concern are the following fractional relaxation inclusions:

$$D_{t,+}^\gamma u(t) \in \mathscr{A} u(t) + f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

and

$$(\mathrm{DFP})_{f,\gamma} : \begin{cases} \mathbf{D}_t^\gamma u(t) \in \mathscr{A} u(t) + f(t),\ t > 0, \\ u(0) = x_0, \end{cases}$$

where $D_{t,+}^\gamma u(t)$ denotes the Weyl–Liouville fractional derivatives of order $\gamma \in (0,1)$, $\mathbf{D}_t^\gamma$ denotes the Caputo fractional derivative of order γ, $x_0 \in X$ and $f : [0,\infty) \to X$ is a given function.

Without specifying particular conditions on function $f(\cdot)$, which will be done later, we will use the following formal definition henceforth:

Definition 1.1. Let $\mathscr{A}$ be a closed MLO.

(i) A continuous function $u : \mathbb{R} \to X$ is said to be a *mild solution of the abstract Cauchy inclusion* of first order

$$u'(t) \in \mathscr{A} u(t) + f(t), \quad t \in \mathbb{R},$$

iff

$$u(t) := \int_{-\infty}^{t} T(t-s) f(s)\, ds, \quad t \in \mathbb{R}.$$

(ii) A continuous function $u : \mathbb{R} \to X$ is said to be a *mild solution of fractional relaxation inclusion* (1.1) iff

$$u(t) = \int_{-\infty}^{t} (t-s)^{\gamma-1} P_\gamma(t-s) f(s)\, ds, \quad t \in \mathbb{R}.$$

(iii) By a mild solution of $(\mathrm{DFP})_{f,\gamma}$, we mean any function $u \in C([0,\infty) : X)$ satisfying that

$$u(t) = S_\gamma(t)x_0 + \int_{0}^{t} R_\gamma(t-s) f(s)\, ds, \quad t > 0.$$

1.2. Stepanov almost periodic functions and asymptotically Stepanov almost periodic functions. Assume that $I = \mathbb{R}$ or $I = [0, \infty)$, and $f : I \to X$ is continuous. Given $\epsilon > 0$, we call $\tau > 0$ an *ϵ-period for $f(\cdot)$* iff

$$\|f(t+\tau) - f(t)\| \leq \epsilon, \quad t \in I.$$

The set consisting of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f, \epsilon)$. A function $f(\cdot)$ is said to be almost periodic, a.p. for short, iff for any $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is relatively dense in I, which means that there exists $l > 0$ such that any subinterval of I of length l meets $\vartheta(f, \epsilon)$. By $AP(I : X)$ we denote the space consisted of all almost periodic functions from the interval I into X.

The class of asymptotically almost periodic functions was introduced by M. Fréchet in 1941. A function $f \in C_b([0, \infty) : X)$ is said to be *asymptotically almost periodic* iff for every $\epsilon > 0$ we can find numbers $l > 0$ and $M > 0$ such that every subinterval of $[0, \infty)$ of length l contains, at least, one number τ such that $\|f(t + \tau) - f(t)\| \leq \epsilon$ for all $t \geq M$. The space consisting of all asymptotically almost periodic functions from $[0, \infty)$ into X is denoted by $AAP([0, \infty) : X)$. For any function $f \in C([0, \infty) : X)$, we have the equivalence of the statements (a)-(c), where:

(i) $f \in AAP([0, \infty) : X)$.

(ii) There exist uniquely determined functions $g \in AP([0, \infty) : X)$ and $\phi \in C_0([0, \infty) : X)$ such that $f = g + \phi$.

(iii) The set $H(f) := \{f(\cdot + s) : s \geq 0\}$ is relatively compact in $C_b([0, \infty) : X)$.

Let $1 \leq p < \infty$, let $l > 0$, and let $f, g \in L^p_{\text{loc}}(I : X)$, where $I = \mathbb{R}$ or $I = [0, \infty)$. The Stepanov 'metric' is defined by

$$D^p_{S_l}[f(\cdot), g(\cdot)] := \sup_{x \in I} \left[\frac{1}{l} \int_x^{x+l} \|f(t) - g(t)\|^p \, dt \right]^{1/p}.$$

Then we know that, for every two numbers $l_1, l_2 > 0$, there exist two positive real constants $k_1, k_2 > 0$ independent of f, g, such that

$$k_1 D^p_{S_{l_1}}[f(\cdot), g(\cdot)] \leq D^p_{S_{l_2}}[f(\cdot), g(\cdot)] \leq k_2 D^p_{S_{l_1}}[f(\cdot), g(\cdot)],$$

as well as that there exists

$$D^p_W[f(\cdot), g(\cdot)] := \lim_{l \to \infty} D^p_{S_l}[f(\cdot), g(\cdot)] \tag{1.2}$$

in $[0, \infty]$. The distance in (1.2) is called the *Weyl distance* of $f(\cdot)$ and $g(\cdot)$. The Stepanov and Weyl 'norm' of $f(\cdot)$ are defined through

$$\|f\|_{S^p_l} := D^p_{S_l}[f(\cdot), 0] \quad \text{and} \quad \|f\|_{W^p} := D^p_W[f(\cdot), 0],$$

respectively.

The choice of length $l > 0$ is basically irrelevant for considerations of Stepanov class. In the sequel, for this class we will assume that $l = 1$. It is said that a function

$f \in L^p_{\rm loc}(I : X)$ is Stepanov p-bounded, S^p-bounded shortly, iff

$$\|f\|_{S^p} := \sup_{t\in I} \left(\int_t^{t+1} \|f(s)\|^p \, ds \right)^{1/p} < \infty.$$

Endowed with the above norm, the space $L^p_S(I : X)$ consisting of all S^p-bounded functions is a Banach space; $L^p_S(I) \equiv L^p_S(I : \mathbb{C})$. A function $f \in L^p_S(I : X)$ is said to be *Stepanov p-almost periodic*, *S^p-almost periodic* shortly, iff the function $\hat{f} : I \to L^p([0,1] : X)$, defined by

$$\hat{f}(t)(s) := f(t+s), \quad t \in I, \ s \in [0,1],$$

is almost periodic. It is said that $f \in L^p_S([0,\infty) : X)$ is *asymptotically Stepanov p-almost periodic*, *asymptotically S^p-almost periodic* shortly, iff $\hat{f} : [0,\infty) \to L^p([0,1] : X)$ is asymptotically almost periodic. Any (asymptotically) Stepanov p-almost periodic function is (asymptotically) equi-Weyl-p-almost periodic.

1.3. Weyl almost periodic functions and asymptotically Weyl almost periodic functions. Let $I = \mathbb{R}$ or $I = [0,\infty)$. The notion of an (equi-)Weyl$p$-almost periodic function is introduced as follows:

Definition 1.2. Let $1 \leq p < \infty$ and $f \in L^p_{\rm loc}(I : X)$.

(i) It is said that the function $f(\cdot)$ is *equi-Weyl-p-almost periodic*, $f \in e\text{-}W^p_{ap}(I : X)$ for short, iff for each $\epsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{x\in I} \left[\frac{1}{l} \int_x^{x+l} \|f(t+\tau) - f(t)\|^p \, dt \right]^{1/p} \leq \epsilon, \text{ i.e., } D^p_{S_l}[f(\cdot+\tau), f(\cdot)] \leq \epsilon.$$

(ii) It is said that the function $f(\cdot)$ is *Weyl-p-almost periodic*, $f \in W^p_{ap}(I : X)$ for short, iff for each $\epsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\lim_{l\to\infty} \sup_{x\in I} \left[\frac{1}{l} \int_x^{x+l} \|f(t+\tau) - f(t)\|^p \, dt \right]^{1/p} \leq \epsilon, \text{ i.e., } \lim_{l\to\infty} D^p_{S_l}[f(\cdot+\tau), f(\cdot)] \leq \epsilon.$$

In the set theoretical sense, we have the following inclusions:

$$APS^p(I : X) \subseteq e\text{-}W^p_{ap}(I : X) \subseteq W^p_{ap}(I : X).$$

As it is well-known, any of these two inclusions can be strict (25). Let us recall that the space of scalar-valued functions $W^p_{ap}(\mathbb{R} : \mathbb{R})$ is introduced for the first time by A. S. Kovanko [26] in 1944.

It is well known that for any function $f \in L^p_{\rm loc}(I : X)$ its Stepanov boundedness is equivalent to its Weyl boundedness, i.e.,

$$\|f\|_{S^p} < \infty \ \text{ iff } \ \|f\|_{W^p} < \infty.$$

For more details about Weyl-almost periodic functions, we refer the reader to [25, Section 4].

Asymptotically (equi)-Weyl-p-almost periodic functions have been recently introduced by the author [6]. If $q \in L^p_{\mathrm{loc}}([0,\infty) : X)$, then we define the function $\mathbf{q}(\cdot,\cdot) : [0,\infty) \times [0,\infty) \to X$ by

$$\mathbf{q}(t,s) := q(t+s), \quad t,s \geq 0.$$

Definition 1.3. It is said that $q \in L^p_{\mathrm{loc}}([0,\infty) : X)$ is *Weyl-p-vanishing* iff

$$\lim_{t\to\infty} \|\mathbf{q}(t,\cdot)\|_{W^p} = 0, \text{ i.e., } \lim_{t\to\infty}\lim_{l\to\infty}\sup_{x\geq 0}\left[\frac{1}{l}\int_x^{x+l}\|q(t+s)\|^p\,ds\right]^{1/p} = 0. \tag{1.3}$$

Replacing the limits in (1.3), we come to the class of equi-Weyl-p-vanishing functions. It is said that a function $q \in L^p_{\mathrm{loc}}([0,\infty) : X)$ is *equi-Weyl-p-vanishing* iff

$$\lim_{l\to\infty}\lim_{t\to\infty}\sup_{x\geq 0}\left[\frac{1}{l}\int_x^{x+l}\|q(t+s)\|^p\,ds\right]^{1/p} = 0.$$

By $W^p_0([0,\infty) : X)$ and e-$W^p_0([0,\infty) : X)$, we denote the vector spaces consisting of all Weyl-p-vanishing functions and equi-Weyl-p-vanishing functions, respectively.

2. Weyl almost periodic and asymptotically Weyl almost periodic solutions to abstract linear Volterra integro-differential equations

We start this section by stating the following proposition:

Proposition 2.1. *Suppose that $1 \leq p < \infty$, $1/p + 1/q = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying that*

$$M := \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < \infty. \tag{2.1}$$

If $g : \mathbb{R} \to X$ is (equi)-Weyl-p-almost periodic, then the function $G : \mathbb{R} \to Y$, given by

$$G(t) := \int_{-\infty}^{t} R(t-s)g(s)\,ds, \quad t \in \mathbb{R}, \tag{2.2}$$

is well-defined, bounded continuous and (equi)-Weyl-p-almost periodic.

Proof. Without loss of generality, we may assume that $X = Y$. We will follow the proof of [24, Proposition 2.11] with appropriate changes. The integral

$$G(t) = \int_0^{\infty} R(s)g(t-s)\,ds$$

is absolutely convergent due to the Hölder inequality, (2.1) and S^p-boundedness of function $g(\cdot)$:

$$\int_0^\infty \|R(s)\|\|g(t-s)\|\,ds = \sum_{k=0}^{\infty} \int_k^{k+1} \|R(s)\|\|g(t-s)\|\,ds$$
$$\leq \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]}\|g\|_{S^p} = M\|g\|_{S^p}, \quad t \geq 0.$$

This also implies the boundedness of function $G(\cdot)$, while its continuity can be proved as in [27, Proposition 5]. It remains to be shown that $G(\cdot)$ is (equi)-Weyl-p-almost periodic. The proof is completely same for both classes and we will show this only for (equi)-Weyl-p-almost periodic functions. So, let a number $\epsilon > 0$ be given. By definition, there exist two finite numbers $l > 0$ and $L > 0$ such that any subinterval I of $\mathbb{R}$ of length L contains a number $\tau \in I$ such that

$$\sup_{x\in\mathbb{R}} \left[\frac{1}{l} \int_x^{x+l} \|f(t+\tau) - f(t)\|^p\,dt \right]^{1/p} \leq \epsilon. \tag{2.3}$$

Arguing as in the proof of [24, Proposition 2.11], we get

$$\|G(t+\tau) - G(t)\| \leq \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \left(\int_{t-k-1}^{t-k} \|g(s+\tau) - g(s)\|^p\,ds \right)^{1/p}, \quad t \geq 0.$$

Using the monotone convergence theorem and this estimate, it readily follows that

$$\sup_{x\in\mathbb{R}} \left[\frac{1}{l} \int_x^{x+l} \|G(t+\tau) - G(t)\|^p\,dt \right]^{1/p}$$
$$\leq \sup_{x\in\mathbb{R}} \left[\frac{1}{l} \int_x^{x+l} \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \left(\int_{t-k-1}^{t-k} \|g(s+\tau) - g(s)\|^p\,ds \right) dt \right]^{1/p}$$
$$= \sup_{x\in\mathbb{R}} \left[\sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \frac{1}{l} \int_x^{x+l} \left(\int_{t-k-1}^{t-k} \|g(s+\tau) - g(s)\|^p\,ds \right) dt \right]^{1/p}$$
$$\leq \left[\sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \left\{ \sup_{x\in\mathbb{R}} \frac{1}{l} \int_x^{x+l} \left(\int_{t-k-1}^{t-k} \|g(s+\tau) - g(s)\|^p\,ds \right) dt \right\} \right]^{1/p}.$$

Hence,

$$\sup_{x\in\mathbb{R}} \left[\frac{1}{l} \int_x^{x+l} \|G(t+\tau) - G(t)\|^p\,dt \right]^{1/p}$$

$$\leq \left[\sum_{k=0}^{\infty}\|R(\cdot)\|_{L^q[k,k+1]}\left\{\sup_{x\in\mathbb{R}}\frac{1}{l}\int_{x-k}^{x-k+1}\int_{x}^{s+k}\|g(s+\tau)-g(s)\|^p\,dt\,ds\right.\right.$$

$$+\frac{1}{l}\int_{x-k+1}^{x+l-k}\int_{s+k-1}^{s+k}\|g(s+\tau)-g(s)\|^p\,dt\,ds$$

$$\left.\left.+\frac{1}{l}\int_{x+l-k}^{x+l-k+1}\int_{s+k-1}^{x+l}\|g(s+\tau)-g(s)\|^p\,dt\,ds\right\}\right]^{1/p} := I+II+III,$$

with the clear meaning of I, II and III. For I and III, the length of segments $[x-k,x-k+1]$ and $[x+l-k,x+l-k+1]$ equals one while the length of segments $[x,s+k]$ and $[s-k+1,x+l]$ does not exceed l. Keeping in mind (2.3), we get

$$I+III\leq 2M\epsilon^p.$$

For any $x\in\mathbb{R}$, we have

$$\frac{1}{l}\int_{x-k+1}^{x+l-k}\int_{s+k-1}^{s+k}\|g(s+\tau)-g(s)\|^p\,dt\,ds$$

$$\leq\frac{1}{l}\int_{x-k+1}^{x+l-k}\|g(s+\tau)-g(s)\|^p\,ds\leq\frac{1}{l}\int_{x-k}^{x+l-k}\|g(s+\tau)-g(s)\|^p\,ds\leq\epsilon^p.$$

Hence $II\leq M\epsilon^p$ and therefore

$$\sup_{x\in\mathbb{R}}\left[\frac{1}{l}\int_{x}^{x+l}\|G(t+\tau)-G(t)\|^p\,dt\right]^{1/p}\leq 3M\epsilon^p.$$

This completes the proof. □

Remark 2.2. Let $t\mapsto\|R(t)\|$, $t\in(0,1]$, be an element of the space $L^q[0,1]$. By the consideration given in [24, Remark 2.12], the condition (2.1) holds in the case that $(R(t))_{t>0}$ is exponentially decaying at infinity or that there exists a finite number $\zeta<0$ such that $\|R(t)\|=O(t^{\zeta})$, $t\to+\infty$, and

(i) $p=1$ and $\zeta<-1$,

or

(ii) $p>1$ and $\zeta<(1/p)-1$.

Remark 2.3. Suppose that $g:\mathbb{R}\to X$ is only Stepanov p-bounded. Then the same argumentation as above shows that the function $G(\cdot)$ given by (2.2) is bounded and continuous.

Example 2.4. It is well known that the function $g(\cdot):=\chi_{[0,1/2]}(\cdot)$, where $\chi_A(\cdot)$ denotes the characteristic function of set A, is (equi)-Weyl-p-almost periodic for any $p\in[1,\infty)$ but not Stepanov almost periodic [25]. If we take $X=Y=\mathbb{C}$ and

$R(t) = e^{-t}$ for $t > 0$, then the function $G(\cdot)$ will be given by $G(t) = 0$ for $t \leq 0$, $G(t) = 1 - e^{-t}$ for $0 \leq t \leq 1/2$ and

$$G(t) = e^{-t} \int\limits_0^{1/2} e^s \, ds$$

for $t > 1/2$. It can be easily seen that this function cannot be Stepanov almost periodic. Similarly, the Heaviside function $g(\cdot) := \chi_{[0,\infty)}(\cdot)$ is W^p_{ap}-almost periodic for any $p \in [1, \infty)$ but not equi-Weyl-1-almost periodic [25]. Taking again $X = Y = \mathbb{C}$ and $R(t) = e^{-t}$ for $t > 0$, the function $G(\cdot)$ will be given by $G(t) = 0$ for $t \leq 0$ and $G(t) = 1 - e^{-t}$ for $t \geq 0$. This function cannot be equi-Weyl-1-almost periodic, so that our result is, in a certain sense, optimal concerning the scale of generalized almost periodic functions. Let us remind ourselves that the Stepanov p-almost periodicity of function $g(\cdot)$ implies that the function $G(\cdot)$ is almost periodic [24].

In the following proposition, we analyze asymptotically (equi)-Weyl-almost periodic properties of finite convolution product (as mentioned in [6], (equi-)Weyl-p-vanishing functions behave much better than Weyl p-pseudo ergodic components introduced by S. Abbas in [28]. In the case that $p > 1$, we basically use the conditions already known from the proof of [24, Proposition 2.13]:

Proposition 2.5. *Suppose that $1 \leq p < \infty$, $1/p + 1/q = 1$ and $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family satisfying that, for every $t \geq 0$, we have*

$$P(t) := \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[t+k,t+k+1]} < \infty.$$

Suppose, further, $p_1 \geq 1$, $P \in W_0^{p_1}([0, \infty) : \mathbb{C})$, resp., $P \in e\text{-}W_0^{p_1}([0, \infty) : \mathbb{C})$, $g : \mathbb{R} \to X$ is (equi)-Weyl-p-almost periodic, and there exists a finite constant $M \geq 1$ such that the function $Q_1(\cdot)$ given by

$$Q_1(t) := \left[\int\limits_M^t \|R(r)\| \|q(t-r)\| \, dr \right]^p, \ t \geq M; \quad Q_1(t) := 0, \quad 0 \leq t \leq M,$$

is in class $W_0^{p_2}([0, \infty) : \mathbb{C})$, resp. $e\text{-}W_0^{p_2}([0, \infty) : \mathbb{C})$, for some number $p_2 \geq 1$ and the function $Q_2(\cdot)$ given by

$$Q_2(t) := \int\limits_{t-M}^t \|q(s)\|^p \, ds, \ t \geq M; \quad Q_2(t) := 0, \quad 0 \leq t \leq M,$$

is in class $W_0^{p_3}([0, \infty) : \mathbb{C})$, resp. $e\text{-}W_0^{p_3}([0, \infty) : \mathbb{C})$, for some number $p_3 \geq 1$. Then the function $H(\cdot)$, given by

$$H(t) := \int\limits_0^t R(t-s)[g(s) + q(s)] \, ds, \quad t \geq 0,$$

is in class $W^p_{ap}([0,\infty):Y)+W^{p_1}_0([0,\infty):Y)+W^{p_2}_0([0,\infty):Y)+W^{p_3}_0([0,\infty):Y)$ *resp.,* $W^p_{eap}([0,\infty):Y)+e\text{-}W^{p_1}_0([0,\infty):Y)+e\text{-}W^{p_2}_0([0,\infty):Y)+e\text{-}W^{p_3}_0([0,\infty):Y)$.

Proof. By Proposition 2.1, $G(\cdot)$ is bounded, continuous and Weyl-p-almost periodic. Define

$$F(t):=\int_0^t R(t-s)q(s)\,ds-\int_t^\infty R(s)g(t-s)\,ds,\quad t\geq 0.$$

Since $H(t)=G(t)+F(t)$ for all $t\geq 0$, $P\in W^{p_1}_0([0,\infty):\mathbb{C})$, resp., $P\in e\text{-}W^{p_1}_0([0,\infty):\mathbb{C})$, $g(\cdot)$ is Stepanov p-bounded and, due to the Hölder inequality,

$$\left\|\int_t^\infty R(s)g(t-s)\,ds\right\|\leq\sum_{k=0}^\infty\int_{t+k+1}^{t+k}\|R(s)\|\|g(t-s)\|\,ds$$
$$\leq\sum_{k=0}^\infty\|R(\cdot)\|_{L^q[t+k,t+k+1]}\|g(t-\cdot)\|_{L^p[t+k,t+k+1]}$$
$$\leq\|g\|_{S^p}\sum_{k=0}^\infty\|R(\cdot)\|_{L^q[t+k,t+k+1]}=\|g\|_{S^p}P(t),\quad t\geq 0,$$

it suffices to show that the function

$$t\to L(t):=\int_0^t R(t-s)q(s)\,ds,\quad t\geq 0,$$

is in class $W^{p_2}_0([0,\infty):Y)+W^{p_3}_0([0,\infty):Y)$, resp., $e\text{-}W^{p_2}_0([0,\infty):Y)+e\text{-}W^{p_3}_0([0,\infty):Y)$. For this, just observe that

$$\left\|\int_0^t R(t-r)q(r)\,dr\right\|\leq\int_0^{t-M}\|R(t-r)\|\|q(r)\|\,dr+\|R(\cdot)\|_{L^q[0,M]}\|q\|_{L^p[t-M,t]},$$

for any $t\geq M$, and employ after that the conditions imposed on the functions $Q_1(\cdot)$ and $Q_2(\cdot)$. □

Remark 2.6. The condition $P\in W^p_0([0,\infty):\mathbb{C})$, resp., $P\in e\text{-}W^p_0([0,\infty):\mathbb{C})$ is slightly weaker from that one employed in the formulation (ii) of [24, Proposition 2.13] since the class of Stepanov p-vanishing functions is contained in the class of equi-Weyl-p-vanishing functions [6]. In the case that $P(t)<\infty$ for all $t\geq 0$ and there exists a finite real number $\zeta<-1$ such that

$$\|R(t)\|=O((1+t^\zeta)^{-1}),\quad t\geq 1,$$

then the stronger condition $P\in C_0([0,\infty):\mathbb{C})$ holds true; see [24, Remark 2.14(ii)].

Remark 2.7. In our previous research study [6], we have used a different condition on the function $q(\cdot)$ provided that $p=1$. For any locally integrable function

$q \in L^1_{\mathrm{loc}}(\mathbb{R} : X)$ and for any strongly continuous operator family $(R(t))_{t>0} \subseteq L(X,Y)$ satisfying

$$\int_0^\infty \|R(s)\|\, ds < \infty,$$

we have formally set

$$J(t,l) := \sup_{x\geq 0} \left\{ \int_0^{x+t} \left[\frac{1}{l} \int_{x+t-r}^{x+t-r+l} \|R(v)\|\, dv \right] \|q(r)\|\, dr \right\}, \quad t>0,\ l>0.$$

In [6], we have used the condition

$$\lim_{t\to\infty} \lim_{l\to\infty} J(t,l) = 0, \quad \text{resp.,} \quad \lim_{l\to\infty} \lim_{t\to\infty} J(t,l) = 0. \tag{2.4}$$

Some sufficient conditions ensuring the validity of (2.4) have been examined in [6, Example 5.4-Example 5.6].

3. Weyl almost periodic and asymptotically Weyl almost periodic solutions to abstract semilinear Volterra integro-differential equations

For the sake of brevity and better expostion, in this section we will use only one pivot space $X = Y$. For our study of semilinear problems, we need the following definition introduced recently in [5], with $I = \mathbb{R}$:

DEFINITION 3.1. A function $f : I \times X \to X$ is said to be equi-Weyl-almost periodic in $t \in I$ uniformly with respect to compact subsets of X iff the function $f(\cdot, x) \in L^p_{\mathrm{loc}}(I : X)$ for each fixed element $x \in X$ and if for each $\epsilon > 0$ there exists $l > 0$ such that for all compacts K of X we have that the set

$$\left\{ \tau \in I : \sup_{u\in K} \sup_{x\in I} \left[\frac{1}{l} \int_x^{x+l} \|f(t+\tau,u) - f(t,u)\|^p\, dt \right]^{1/p} < \epsilon \right\}$$

is relatively dense. We denote by $e\text{-}W^p_{ap,K}(I \times X : X)$ the vector space consisting of such functions.

A useful characterization of equi-Weyl-p-almost periodic functions established by L. I. Danilov [29] has been essentially employed by F. Bedouhene, Y. Ibaouene, O. Mellah and P. Raynaud de Fitte [5] for proving the following composition principle with $I = \mathbb{R}$ (see [5, Theorem 3]); the proof also works in the case that $I = [0, \infty)$:

Lemma 3.2. *Suppose that $p, q, r \geq 1$, $1/r + 1/q = 1/p$ and $f(\cdot,\cdot) \in e\text{-}W^p_{ap,K}(I \times X : X)$. If there exists a Stepanov r-bounded function $L(\cdot)$ such that*

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x-y\|, \quad t \in I,\ x,\ y \in X, \tag{3.1}$$

then for each function $x \in e\text{-}W^q_{ap}(I : X)$ *we have* $f(\cdot, x(\cdot)) \in e\text{-}W^p_{ap}(I : X)$.

The composition principles for Weyl-p-almost periodic functions will be considered somewhere else and, in this section, we will consider only the class of equi-Weyl-p-almost periodic functions (it is also worth mentioning an interesting result of S. Abbas [28], concerning composition principles for the class of Weyl pseudo almost automorphic functions).

We need to introduce the following definition:

DEFINITION 3.3. Let $q : [0,\infty) \times X \to X$ be such that $q(\cdot, x) \in L^p_{\text{loc}}([0,\infty) : X)$ for each fixed element $x \in X$.

(i) We say that $q(\cdot,\cdot)$ is *Weyl-p-vanishing uniformly with respect to compact subsets of* X iff for each compact set K of X we have:

$$\lim_{t\to\infty} \lim_{l\to\infty} \sup_{\xi\geq 0, x\in K} \left[\frac{1}{l}\int_{\xi}^{\xi+l} \|q(t+s,x)\|^p\, ds\right]^{1/p} = 0. \tag{3.2}$$

(ii) We say that $q(\cdot,\cdot)$ is *equi-Weyl-p-vanishing uniformly with respect to compact subsets of* X iff for each compact set K of X we have:

$$\lim_{l\to\infty} \lim_{t\to\infty} \sup_{\xi\geq 0, x\in K} \left[\frac{1}{l}\int_{\xi}^{\xi+l} \|q(t+s,x)\|^p\, ds\right]^{1/p} = 0. \tag{3.3}$$

We denote by $W^p_{0,K}(I \times X : X)$ and $e\text{-}W^p_{0,K}(I \times X : X)$ the classes consisting of all Weyl-p-vanishing functions, uniformly with respect to compact subsets of X and equi-Weyl-p-vanishing functions, uniformly with respect to compact subsets of X, respectively.

Now we are ready to state the following composition principle:

Theorem 3.4. *Suppose that*

$$p, q, r \geq 1, \quad \frac{1}{r} + \frac{1}{q} = \frac{1}{p}, \quad g(\cdot,\cdot) \in e\text{-}W^p_{ap,K}([0,\infty) \times X : X)$$

and there exists a Stepanov r*-bounded function* $L(\cdot)$ *such that* (3.1) *holds with the function* $f(\cdot,\cdot)$ *replaced with the function* $g(\cdot,\cdot)$ *therein. Suppose, further, that the following conditions hold:*

(i) *The function* $Q := f - g : [0,\infty) \times X \to X$ *is Weyl-p-vanishing uniformly with respect to compact subsets of* X*, resp., equi-Weyl-p-vanishing uniformly with respect to compact subsets of* X*. Denote by* $W^p_{0,Q}([0,\infty) : X)$ *the class* $W^p_0([0,\infty) : X)$*, resp.* $e\text{-}W^p_0([0,\infty) : X)$*, in the first, resp., the second case.*

(ii) *The function* $z : [0,\infty) \to X$ *is Weyl-q-vanishing, resp., equi-Weyl-q-vanishing. Denote by* $W^q_{0,z}([0,\infty) : X)$ *the class* $W^q_0([0,\infty) : X)$*, resp.* $e\text{-}W^q_0([0,\infty) : X)$*, in the first, resp., the second case.*

(iii) *The function* $y : [0,\infty) \to X$ *is equi-Weyl-q-almost periodic and there exists a set* $E' \subseteq [0,\infty)$ *with* $m(E') = 0$ *such that* $K' = \{x(t) : t \in [0,\infty) \setminus E'\}$

is relatively compact in X, where $m(E')$ denotes the Lebesgue measure of E' and $x(\cdot) := y(\cdot) + z(\cdot)$.

Then the mapping $t \mapsto f(t, x(t))$, $t \geq 0$ is in class e-$W^p_{ap}([0,\infty) : Y) + W^p_{0,Q}([0,\infty) : X) + W^q_{0,z}([0,\infty) : X)$.

Proof. It is clear that

$$f(t, x(t)) = [g(t, x(t)) - g(t, y(t))] + g(t, y(t)) + Q(t, x(t)), \quad t \geq 0.$$

Due to Lemma 3.2, it suffices to show that the function $g(\cdot, x(\cdot)) - g(\cdot, y(\cdot))$ is in class $W^q_{0,z}([0,\infty) : X)$ and the function $Q(\cdot, x(\cdot))$ is in class $W^p_{0,Q}([0,\infty) : X)$. This is evident for the second function since we have assumed (iii) and (3.2), resp., (3.3) holds. For the first function, the corresponding statement holds on account of the fact that $p > q$, the estimate

$$\|g(t, x(t)) - g(t, y(t))\| \leq L(t)\|z(t)\|, \quad t \geq 0,$$

and the following calculation involving the Hölder inequality:

$$\begin{aligned}
\left[\frac{1}{l}\int_{x+t}^{x+t+l} \|g(s, x(s)) - g(s, y(s))\|^p\, ds\right]^{1/p} &\leq \left[\frac{1}{l}\int_{x+t}^{x+t+l} [L(s)\|z(s)\|]^p\, ds\right]^{1/p} \\
&\leq l^{(-1)/p}\left[\sum_{k=0}^{\lfloor l\rfloor} \|L\|_{L^r[x+t+k,x+t+k+1]}\|z\|_{L^q[x+t+k,x+t+k+1]}\right]^{1/p} \\
&\leq l^{(-1)/p}\|L\|_{S^r}\left[\sum_{k=0}^{\lfloor l\rfloor} \|z\|_{L^q[x+t+k,x+t+k+1]}\right]^{1/p} \\
&\leq l^{(-1)/p}\|L\|_{S^r} l^{1/p-1/q}\left[\sum_{k=0}^{\lfloor l\rfloor} \|z\|^q_{L^q[x+t+k,x+t+k+1]}\right]^{1/q} \\
&= \|L\|_{S^r} l^{(-1)/q}\|z\|_{L^q[x+t,x+t+l]}. \quad \square
\end{aligned}$$

In the remaining part of this section, we will consider abstract semilinear Cauchy problems. First of all, we will state and prove the following slight generalization of the first part of [5, Theorem 5], stated only in the case that $p \geq 2$ (see also [5, Remark 2], where the authors have imposed the constant value for the function $L(\cdot)$):

Theorem 3.5. *Suppose that $1 \leq p < \infty$, $1/p + 1/q = 1$ and $(R(t))_{t>0} \subseteq L(X)$ is a strongly continuous operator family satisfying* (2.1). *Suppose that the function $t \mapsto f(t, v(t))$, $t \in \mathbb{R}$ is locally p-integrable for any function $v \in C_b(\mathbb{R} : X)$ and there exists a function $L \in L^p_S(\mathbb{R})$ such that* (3.1) *holds with $I = \mathbb{R}$. If $M\|L\|_{S^p} < 1$, then there exists a unique function $u \in C_b(\mathbb{R} : X)$ such that*

$$u(t) := \int_{-\infty}^{t} R(t - s) f(s, u(s))\, ds, \quad t \in \mathbb{R}. \tag{3.4}$$

PROOF. Set, for every function $v \in C_b(\mathbb{R} : X)$,

$$(\Upsilon v)(t) := \int_{-\infty}^{t} R(t-s)f(s,v(s))\,ds, \quad t \in \mathbb{R}.$$

We claim that the mapping $\Upsilon : C_b(\mathbb{R} : X) \to C_b(\mathbb{R} : X)$ is well-defined and an $(M\|L\|_{S^p})$-contraction. Since the function $t \mapsto f(t,v(t))$, $t \in \mathbb{R}$ is locally p-integrable for any function $v \in C_b(\mathbb{R} : X)$, we may use the same argumentation as in the proof of [27, Proposition 5] in order to see that the function $(\Upsilon v)(\cdot)$ is continuous. For boundedness, observe first that the mapping $f(\cdot,0)$ is S^p-bounded since $\{0\}$ is a compact subset of X. After that, we can employ the following calculation, which is valid for each number $t \in \mathbb{R}$:

$$\begin{aligned}
\left\| \int_{-\infty}^{t} R(t-s)f(s,v(s))\,ds \right\| &\leq \sum_{k=0}^{\infty} \int_{t-(k-1)}^{t-k} \|R(t-s)\| \|f(s,v(s))\|\,ds \\
&\leq \sum_{k=0}^{\infty} \int_{t-(k-1)}^{t-k} \|R(t-s)\| [\|f(s,v(s))\| - \|f(s,0)\|]\,ds \\
&\quad + \sum_{k=0}^{\infty} \int_{t-(k-1)}^{t-k} \|R(t-s)\| \|f(s,0)\|\,ds \\
&\leq \sum_{k=0}^{\infty} \|R(t-\cdot)\|_{L^q[t-(k-1),t-k]} \|L(\cdot)v(\cdot)\|_{L^p[t-(k-1),t-k]} \\
&\quad + \|f(\cdot,0)\|_{S^p} \sum_{k=0}^{\infty} \int_{t-(k-1)}^{t-k} \|R(t-\cdot)\|_{L^q[t-(k-1),t-k]} \\
&\leq M\|L\|_{S^p}\|v\|_\infty + M\|f(\cdot,0)\|_{S^p}.
\end{aligned}$$

Arguing similarly (see also the proof of [4, Theorem 2.7.6]), we can conclude that for every two functions $v_1, v_2 \in C_b(\mathbb{R} : X)$ one has

$$\|(\Upsilon v_1)(\cdot) - (\Upsilon v_2)(\cdot)\|_\infty \leq M\|L\|_{S^p}\|v_1 - v_2\|_\infty,$$

proving the claimed. $\square$

In the following theorem, we consider abstract semilinear Cauchy problems for functions defined on the non-negative real axis:

Theorem 3.6. *Suppose that $1 \leq p < \infty$, $1/p+1/q = 1$ and $f : [0,\infty) \times X \to X$ satisfies that the function $t \mapsto f(t,v(t))$, $t \geq 0$ is locally p-integrable for any function $v \in C_b([0,\infty) : X)$. Suppose, further, $x \in E$ and $(R(t))_{t>0} \subseteq L(X)$ is a strongly continuous operator family satisfying* (2.1)*. Let there exist a function $L \in L^p_S([0,\infty))$ such that* (3.1) *holds with $I = [0,\infty)$. If $M\|L\|_{S^p} < 1$ and the mapping $t \mapsto R(t)x$, $t > 0$, can be extended to a bounded continuous function defined on $[0,\infty)$, denoted* *by the same symbol henceforth, then there exists a unique function* $u \in C_b([0,\infty) : X)$ *such that*

$$u(t) := R(t)x + \int_0^t R(t-s)f(s,u(s))\,ds, \quad t \geq 0.$$

Proof. The proof is very similar to the proof of previous theorem and we will only outline the main details. Set, for every function $v \in C_b([0,\infty) : X)$,

$$(\Upsilon v)(t) := R(t)x + \int_0^t R(t-s)f(s,u(s))\,ds, \quad t \geq 0.$$

Then the mapping

$$\Upsilon : C_b([0,\infty) : X) \to C_b([0,\infty) : X)$$

is well-defined. For the continuity of $(\Upsilon v)(\cdot)$, we can use the same argument as above. The boundedness of function $(\Upsilon v)(\cdot)$ can be deduced as in the proof of Theorem 3.5 and we have the estimate:

$$\|(\Upsilon v)(t)\| \leq \|R(t)x\| + (\|v\|_\infty + \|f(\cdot,0)\|_{S^p})M < \infty, \quad t \geq 0.$$

It is almost trivial to prove that $\Upsilon(\cdot)$ is an $(M\|L\|_{S^p})$-contraction, finishing the proof of theorem by the Banach contraction principle. $\square$

It is a well-known fact that Stepanov-p-almost periodicity of function $f(\cdot,\cdot)$ in Theorem 3.5, resp. asymptotical Stepanov-p-almost periodicity of function $f(\cdot,\cdot)$ in Theorem 3.6, almost inevitably leads to the almost periodicity, resp. asymptotical almost periodicity, of obtained solutions (cf. [4] for more details). On the other hand, the spaces of (equi)-Weyl-p-almost periodic functions are not complete with respect to the Weyl norm and the above conclusion cannot be deduced. Following the method proposed by M. Kamenskii, O. Mellah and P. Raynaud de Fitte [30], the authors of [5] have investigated some additional conditions ensuring that the obtained bounded continuous solution of the abstract semilinear Cauchy problem under their consideration is equi-Weyl-p-almost periodic for a given equi-Weyl-p-almost periodic forcing term $f(\cdot,\cdot) \in e\text{-}W^p_{ap,K}(\mathbb{R} \times X : X)$ (cf. [5, Theorem 5]). It is beyond the scope of this paper to analyze a similar question for degenerate resolvent operator families $(R(t))_{t>0}$ for which the estimate $\|R(t)\| = O(e^{-ct}t^{\beta-1})$, $t > 0$, or $\|R(t)\| = O(t^{\beta-1}(1+t^\gamma)^{-1})$, $t > 0$, holds with certain numbers $c > 0$, $\beta \in (0,1)$ and $\gamma > 1$.

4. Examples and applications

As illustrated in the final section of paper [6], a great number of examples and applications of our abstract results can be given to the class consisting of those multivalued linear operators $\mathscr{A}$ satisfying the condition (P). First of all, it is pretty clear how one can apply Proposition 2.1 and Proposition 2.5 in the qualitative analysis of solutions to abstract degenerate differential equations and inclusions of first order. Typical examples are the Poisson heat equations

$$\frac{\partial}{\partial t}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x), t \in \mathbb{R},\ x \in \Omega;$$
$$v(t,x) = 0, (t,x) \in [0,\infty) \times \partial\Omega,$$

and

$$\frac{\partial}{\partial t}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x), \quad t \geq 0,\ x \in \Omega;$$
$$v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega,$$
$$m(x)v(0,x) = u_0(x), \quad x \in \Omega,$$

in the space $X := L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geq 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$ and $1 < p < \infty$. The starting point in the whole analysis is the fact that the multivalued linear operator $\mathscr{A} := AB^{-1}$, where $A := \Delta - b$ acts on X with the Dirichlet boundary conditions, and B is the multiplication operator by the function $m(x)$, satisfies the condition (P) with $\beta = 1/p$ and certain finite constants $c,\ M > 0$. Proposition 2.1 and Proposition 2.5 can be applied in the analysis of existence and uniqueness of (equi)-Weyl-p-almost periodic solutions and asymptotically (equi)-Weyl-p-almost periodic solutions of the fractional Poisson heat equations containing Weyl-Liouville of Caputo fractional derivatives, as well.

The afore-mentioned results can be also applied to abstract non-degenerate differential equations with almost sectorial operators [31]. For example, we can apply Proposition 2.5 in the analysis of existence and uniqueness of asymptotically Weyl-p-almost periodic solutions of the following fractional equation in the Hölder space $X = C^\alpha(\overline{\Omega})$:

$$\mathbf{D}_t^\gamma u(t,x) = -\sum_{|\beta|\leq 2m} a_\beta(t,x) D^\beta u(t,x) - \sigma u(t,x) + f(t,x), \quad t \geq 0,\ x \in \Omega,$$
$$u(0,x) = u_0(x), \quad x \in \Omega.$$

For more details, the interested reader may consult [32] and [6].

We close the paper with the observation that the assertions of Proposition 2.1 and Proposition 2.5 seem to be new even for non-degenerate strongly continuous exponentially decaying semigroups.

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