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NUMERICAL METHOD FOR SOLVING BOUNDARY INVERSE PROBLEM FOR ONE–DIMENSIONAL PARABOLIC EQUATION

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Abstract. We consider a numerical method for solving boundary inverse problem using the implicit difference scheme for approximation by time and finite difference method for the boundary inverse problem. A numerical solution to the boundary inverse problem is determined by special decomposition which transforms the problem into two standard problems. We present the results of numerical experiments, including those with random errors in the input data, which confirm the capabilities of the proposed computational algorithms for solving this boundary inverse problem.

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1. Introduction

Boundary value problems for partial differential equations are often considered in some applied problems [1, 2]. It is very important to develop and examine some numerical methods for boundary value problems in the field of mathematical physics [3, 4]. Usually solutions to boundary value problems are found from the equations and some additional conditions. For time-independent problems, the boundary conditions are given [5], and for time-dependent equations, initial conditions are needed as well [6]. These boundary value problems are known as the traditional direct problems.

In this paper, we do not consider the direct problem; instead we focus on the inverse problem. Inverse problems are different from the direct ones. In the inverse problem, the master equation, initial and boundary conditions are not fully specified, but some additional information is available [7]. We find the unknown values by using the additional information.

There are many kinds of inverse problems, such as evolutionary inverse problems in which initial conditions are unknown [8], boundary inverse problems in which boundary conditions are unknown [9] and coefficient inverse problems in which the equation is not specified completely as some equation coefficients are unknown

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[10, 11]. The inverse problem which we concentrate on in this article is a boundary inverse problem.

Boundary inverse problems play an important role among the inverse problems of mathematical physics. They are related to diagnosing problems where one needs to restore the boundary regime on some border by using additional measurements on some part of the borders or inside the computation domain. The boundary inverse problem is studied in many works [12, 13].

Inverse problems in mathematical physics are often considered to be a class of classic ill-posed problems due to the lack of dependency of the solution on the input data [14–16]. So most standard numerical methods cannot provide good accuracy; therefore, their numerical solution requires development of special computational algorithms.

In this paper, we consider the boundary inverse problems for the one-dimensional second order parabolic partial differential equation, which is called heat conduction equation, of the following form:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 \le t \le T,\ 0 \le x \le l. \tag{1}$$

In this problem, we need to reconstruct the boundary condition from the measurements which are taken inside the calculation domain. This problem belongs to the class of conditionally well-posed problems [7, 17, 18].

To solve the boundary inverse problem for the heat conduction equation, space approximation is performed with the use of standard finite difference, while linearized time approximations are constructed by the fully implicit scheme. Linear problems at a particular time level are solved by a special decomposition.

2. The Problem Formulation

We consider the problem on domain $[0, l]$. The boundary inverse problem is formulated as follows

$$u_t - u_{xx} = 0, \quad 0 < t \le T,\ x \in [0, l], \tag{2}$$

with the initial condition

$$u(x, 0) = u^0(x), \quad x \in [0, l], \tag{3}$$

In the inverse problem, not all boundary conditions are known; instead, we have the solution at some internal point x^* and the boundary condition on the left boundary, while the conditions are given as

$$u(0, t) = q(t),\ t \in (0, T], \quad u(x^*, t) = g(t),\ x^* \in (0, l),\ t \in (0, T]. \tag{4}$$

The boundary inverse problem is to find solution $u(l, t)$ to the problems (2)–(4). There are many works that study the boundary inverse problems. Some monographs summarize the construction and justification of numerical methods to solve the boundary inverse problems [7, 19]. In this paper, we consider only the numerical solution to the boundary inverse problem of the second order in the heat conduction problem without some theoretical issues.

3. The computational algorithm

We consider the inverse problem (2)–(4) with $\overline{\Omega} = [0, l]$. To solve the parabolic problem numerically, we introduce the grid on domain Ω

$$\omega_h = \{x_i \mid x_i = ih,\ i = 1, 2, \ldots, m, \ldots, M-1,\ mh = x^*,\ Mh = l\},$$

where M is a positive integer. For the time we have

$$\overline{\omega}_\tau = \{t^n \mid t^n = n\tau,\ n = 0, 1, \ldots, N,\ N\tau = T\}.$$

Using the notation $u^n(x) = u(x, t^n)$ and backward difference for approximation in time, we get

$$u_t(x, t^n) \approx \frac{u^n(x) - u^{n-1}(x)}{\tau}, \quad n = 1, 2, \ldots, N.$$

By employing finite difference approximations in space with the new notation $u_i(t) = u(x_i, t)$, we obtain

$$\frac{\partial^2 u}{\partial x^2}(x_i, t) \approx \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{h^2}, \quad i = 1, 2, \ldots, M-1.$$

Introducing symbols $u_i^n = u(x_i, t^n)$, for the equation (2) we have

$$\frac{u_i^n - u_i^{n-1}}{\tau} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}, \tag{5}$$

where $n = 1, 2, \ldots, N$ and $i = 1, 2, \ldots, M-1$. Simplifying (5), we get

$$u_{i+1}^n - \left(2 + \frac{h^2}{\tau}\right) u_i^n + u_{i-1}^n + \frac{h^2}{\tau} u_i^{n-1} = 0, \tag{6}$$

with $u_0^n = q^n$.

We use the following decomposition for the solution u^n at the new time level

$$u_i^n = v_i^n + u_M^n w_i^n, \quad i = 0, 1, \ldots, M, \tag{7}$$

where $v_0^n = q^n$, $v_M^n = 0$ and $w_0^n = 0$, and $w_M^n = 1$. Substituting (7) in (6), we employ the following equations

$$v_{i+1}^n - \left(2 + \frac{h^2}{\tau}\right) v_i^n + v_{i-1}^n + \frac{h^2}{\tau} u_i^{n-1} = 0, \tag{8}$$

$$w_{i+1}^n - \left(2 + \frac{h^2}{\tau}\right) w_i^n + w_{i-1}^n = 0. \tag{9}$$

To calculate u_M^n, we find v^n and w^n from (8) and (9). For the solutions v_i^n and w_i^n, we use the following decompositions

$$v_i^n = \alpha_i^n v_{i+1}^n + \beta_i^n, \quad i = M-1, M-2, \ldots, 0, \tag{10}$$

with $\alpha_0^n = 0$, $\beta_0^n = q^n$ and

$$w_i^n = \alpha_i^n w_{i+1}^n + \gamma_i^n, \quad i = M-1, M-2, \ldots, 0, \tag{11}$$

with $\gamma_0^n = 0$.

Applying (10) to (8), we obtain

$$\alpha_i^n = \frac{1}{2 + \kappa - \alpha_{i-1}^n}, \tag{12}$$

$$\beta_i^n = \frac{\kappa u_i^{n-1} + \beta_{i-1}^n}{2 + \kappa - \alpha_{i-1}^n}, \tag{13}$$

where $\kappa = \frac{h^2}{\tau}$, $i = 1, 2, \ldots, M-1$ and $n = 1, 2, \ldots, N$. Using (11) in (9), we have

$$w_i^n = \alpha_i^n w_{i+1}^n, \quad \gamma_i^n = 0, \tag{14}$$

with $w_M^n = 1$ and $i = M-1, M-2, \ldots, 1$. From (12)–(14), we can obtain all coefficients α, β and γ, so we get all the solutions of v and w.

To evaluate u_M^n, the additional condition $u(x^*, t) = g(t)$ is used. Applying (7) to this additional condition, we have

$$g^n = v^n(x^*) + u_M^n w^n(x^*), \tag{15}$$

where $x^* = mh \in (0, l)$. Thus we get

$$u_M^n = \big(g^n - v_m^n\big)/w_m^n, \tag{16}$$

to solve this equation. We assume that $w_m^n \neq 0$ and v^n and w^n are determined from (8), (9).

To guarantee the correctness of the computational algorithm for solving this inverse problem, the focus of the algorithm is related to the condition $w_m^n \neq 0$. Taking note of that, all the values w_i^n are determined by the system (9), (12) and (14). From (12) we can easily find that $0 < \alpha_1^n < 1$. By the use of mathematical induction we can easily get all $w_i \in (0, 1]$, i. e. all the $w_i > 0$, and guarantee the condition $w_m^n \neq 0$. It should be noted that the value α is associated with $\kappa = \frac{h^2}{\tau}$ and w_i is also affected. So we can control the value of w_m by choosing the appropriate time and space step size in order to make sure that the value of w_m is not too small.

4. Numerical examples

In this section we present numerical results to test the efficiency of the computational algorithm mentioned previously for solving the boundary inverse problems for one-dimensional second order parabolic equations.

To evaluate the numerical errors, we adopt the errors $\epsilon(t)$ which are defined as follows:

$$\epsilon(t) = \tilde{u}(l, t) - u(l, t),$$

where $\tilde{u}(l, t)$ is the numerical solution of $u(l, t)$ at time t, $u(l, t)$ is the exact boundary solution.

Example 1. We consider the boundary inverse problem from the monograph [7] with $l = 1$ and $T = 1$. The initial condition is given as

$$u^0(x) = 0, \quad 0 \leq x \leq l.$$

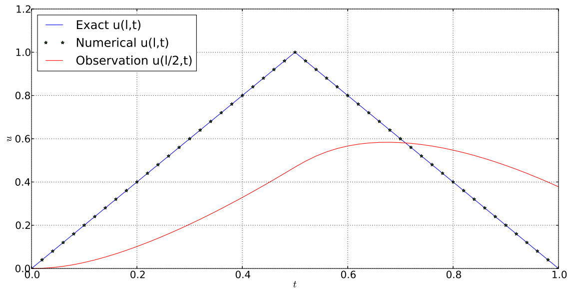


Fig. 1. The exact solution, numerical and the observation solutions

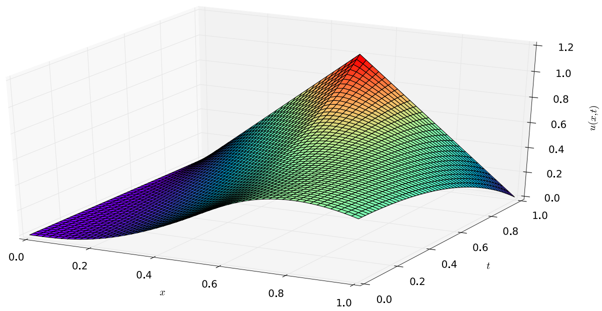


Fig. 2. The space-time figure of numerical solutions $u(x,t)$

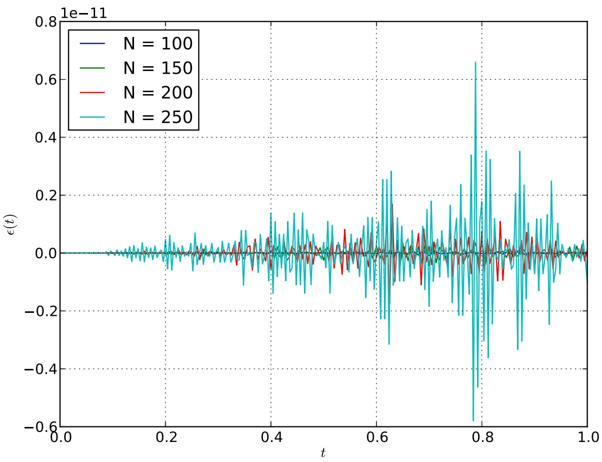


Fig. 3. The errors of $\epsilon(t)$ with different N

The boundary solutions $u(1,t)$ have the following forms:

$$u(1,t) = \begin{cases} \frac{2t}{T}, & 0 \le t < \frac{T}{2}, \\ \frac{2(T-t)}{T}, & \frac{T}{2} \le t \le T. \end{cases}$$

In the computation, we choose $x^* = 0.5$, $N = 50$, and $M = 50$. The exact and numerical solutions $u(l,t)$ and the observation solution at $x^* = 0.5$ with $T = 1.0$ are presented in Fig. 1. The space-time figure of numerical solutions $u(x,t)$ are presented in Fig. 2.

We consider this boundary inverse problem with the observation point $x^* = 0.5$ and $M = 50$ for different N. The dependence of the errors $\epsilon(t)$ on different N is given in Fig. 3.

In this numerical results, we also pay attention to the errors for different M. In the computation, we choose $N = 100$ and the observation point to be $x^* = 0.5$. The dependence of the errors $\epsilon(t)$ on different M is given in Fig. 4.

Finally, we consider the errors $\epsilon(t)$ with different observation points choosing $M = 50$ and $N = 100$. The dependence of the errors $\epsilon(t)$ with different observation points is given in Fig. 5.

Example 2. This test case is taken from [12, 14]. We also put $x \in [0,1]$ with the conditions

$$u^0(x) = 0, \quad 0 \le t \le 1.$$

The boundary solutions $u(1,t)$ take the form

$$u(1,t) = \begin{cases} 0, & 0 \le t < 0.25 \text{ and } 1 \ge t > 0.75, \\ 1, & 0.25 \le t \le 0.75. \end{cases}$$

We consider the inverse problem with $M = 50$ and $N = 50$. The exact and numerical solutions $u(1,t)$ with different observation solutions at x^* are presented in Fig. 6. The space-time dependence of numerical solutions $u(x,t)$ is presented in Fig. 7.

In the numerical algorithm for the inverse problem, the focal point is disturbance of the input data. Much attention is paid to the cases in which input data is given with an error. In our boundary inverse problem while observing the solution at an interior point $u(x^*,t) = g(t)$, $0 < t \le T$, the fundamental problem is to study the influence of inaccuracies in the specified function $g(t)$ on the accuracy of the identification of the boundary solutions.

To illustrate the effectiveness of the numerical method with noisy input data, we introduce the error in the values of the function $g(t)$ at the points of the time grid. We put

$$g_\delta(t^n) = g(t^n) + \delta\sigma, \quad t^n = n\tau, \quad n = 1, 2, \dots, N_1,$$

where δ is the tolerated noise level and σ is a random variable uniformly distributed on the interval $[-0.5, 0.5]$.

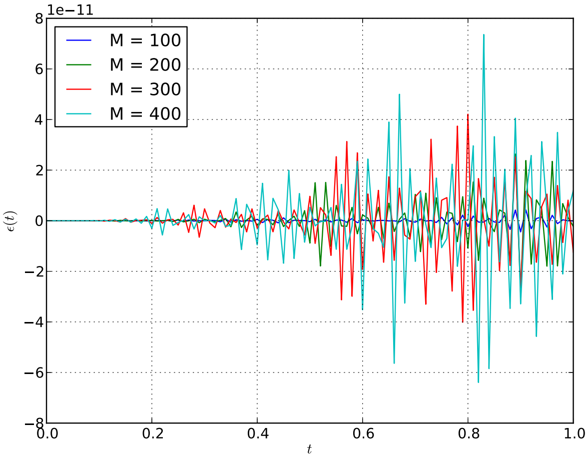


Fig. 4. The errors of $\epsilon(t)$ with different M

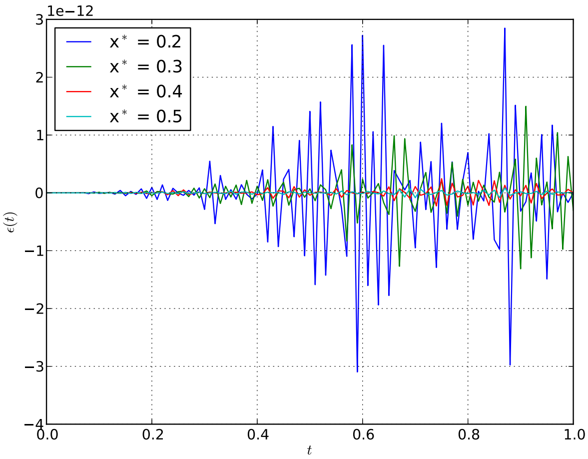
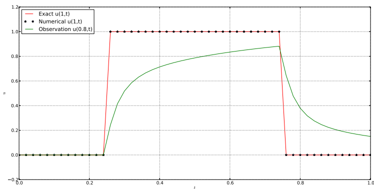


Fig. 5. The errors of $\epsilon(t)$ with different observation points x^*

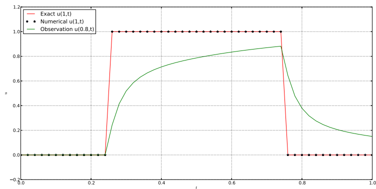
Fig. 8 shows the exact, numerical solutions of the boundary solutions and the solution with noisy data for solving the boundary inverse problem with $\delta = 0.01$. For smoothing the function $g_\delta(t^n)$, $n = 1, 2, \ldots, N_1$, we use the cubic-splines interpolation. In the computation, we choose $M = 50$ and $N_1 = 10$.

We also consider the solutions of the boundary inverse problem with different observation points x^* and $\delta = 0.01$. The dependence of the solution to the boundary on different observation points x^* is given in Fig. 9.

Finally, we consider solutions with different tolerated noise level δ. The solutions with different δ are given in Fig. 10. In this computation, the observation point is



(a) Solutions with observation point $x^* = 0.5$



(b) Solutions with observation point $x^* = 0.8$

Fig. 6. The exact and numerical solutions $u(1,t)$ and observation solutions $u(x^*,t)$

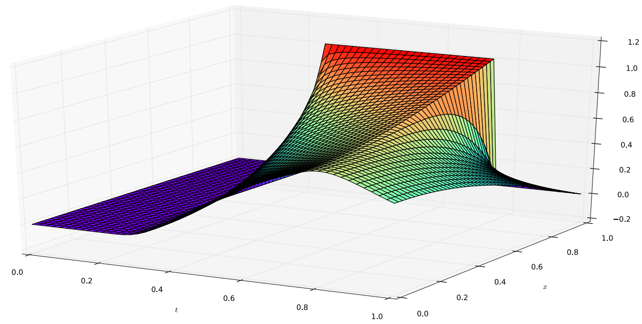


Fig. 7. The space-time figure of numerical solutions $u(x,t)$

chosen to be $x^* = 0.5$. From Fig. 10 we see that when the noisy reaches 10^{-1}, the difference between the exact and numerical solutions increases.

5. Conclusions

We proposed a numerical scheme to solve the inverse problems by the means of the finite difference method. The numerical implementation used the linearized approximations in time and standard finite difference procedures in space based on a decomposition of the approximate solution. Numerical solutions to the model problem demonstrate the convergence of the approximate solutions to the inverse problem.

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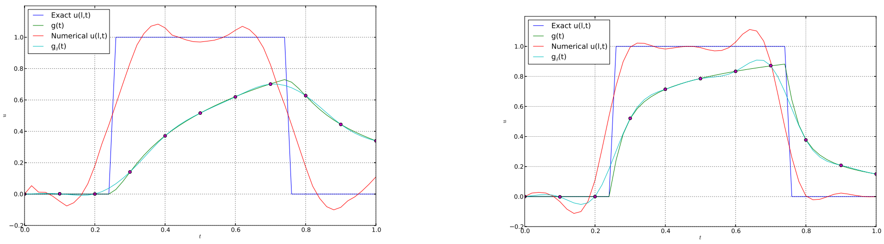
REFERENCES

1. *Olver P.*, Introduction to Partial Differential Equations, Springer, Cham (2014).

2. *Guenther R. B. and Lee J. W.*, Partial Differential Equations of Mathematical Physics and Integral Equations, Dover Publ., New York (2012).



(a) Solutions with observation point $x^* = 0.5$ (b) Solutions with observation point $x^* = 0.8$

Fig. 8. The exact and numerical solutions $u(1,t)$ and noisy data $g_\delta(t)$

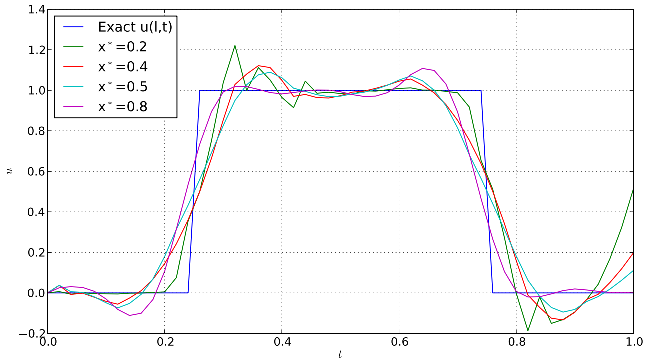


Fig. 9. The solutions with different observation point x^*

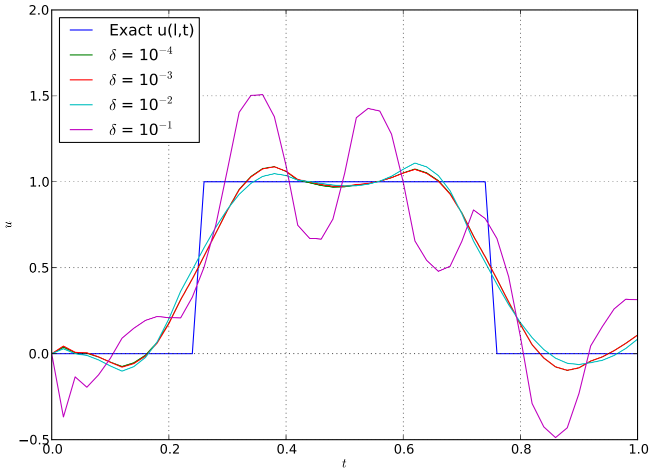


Fig. 10. The exact and numerical solution with different δ

3. *Tikhonov A. N. and Samarskii A. A.*, Equations of Mathematical Physics, Dover Publ., New York (2011).

4. *Evans L. C.*, Partial Differential Equations, Amer. Math. Soc., Washington (2010).

5. *Gilbarg D. and Trudinger N. S.*, Elliptic Partial Differential Equations of Second Order, Springer, Berlin, Heidelberg (2001).

6. *Jiang Z. W.*, "A meshfree method for numerical solution of nonhomogeneous time dependent problems," Abstract Appl. Anal., Article ID 978310 (2014).

7. *Samarskii A. A. and Vabishchevich P. N.*, Numerical Methods for Solving Inverse Problems of Mathematical Physics, Walter de Gruyter, Berlin (2007).

8. *Jiang T. S., Li M., and Chen C. S.*, "The method of particular solutions for solving inverse problems for a nonhomogeneous convection diffusion equation with variable coefficients," Numer. Heat Transfer, P. A: Applications, **61**, 338–352 (2012).

9. *Kang H.*, "A uniqueness theorem for an inverse boundary value problem in two dimensions," J. Math. Anal. Appl., **270**, No. 1, 291–302 (2002).

10. *Vabishchevich P. N. and Vasil'ev V. I.*, "Computational algorithms for solving the coefficient inverse problem for parabolic equations," Inverse Probl. Sci. Engin., **24**, No. 1, 42–59 (2016).

11. *Vabishchevich P. N.*, "Iterative computational identification of a spacewise dependent the source in a parabolic equations," Inverse Probl. Sci. Engin., **25**, No. 8, 1168–1190 (2017).

12. *Vasil'ev V. V., Vasilyeva M. V., and Kardashevsky A. M.*, "The numerical solution of the boundary inverse problem for a parabolic equation," in: Application of Mathematics in Technical and Natural Sciences, AIP Conf. Proc., AIP Publ., **1773**, No. 1, 100010 (2016).

13. *Colaço M. J., Orlande H. R. B., and Dulikravich G. S.*, "Inverse and optimization problems in heat transfer," J. Braz. Soc. Mech. Sci. Engin., **28**, No. 1/1 (2006).

14. *Kamynin V. L.*, "On the solvability of the inverse problem for determining the right hand side of a degenerate parabolic equation with integral observation," Math. Notes, **98**, No. 5, 765–777 (2015).

15. *Kamynin V. L.* , "The inverse problem of the simultaneous determination of the right hand side and the lowest coefficient in parabolic equations with many space variables," Math. Notes, **97**, No. 3, 349–361 (2015).

16. *Vabishchevich P. N., Vasil'ev V. I., and Vasil'eva M. V.*, "Computational identification of the right hand side of a parabolic equation," Comput. Math. Math. Phys., **55**, No. 9, 1015–1021 (2015).

17. *Kabanikhin S. I.*, Inverse and Ill-Posed Problems: Theory and Applications, Walter de Gruyter, (2011).

18. *Jonas P. and Louis A. K.*, "Approximate inverse for a one-dimensional inverse heat conduction problem," Inverse Problems, **16**, 175–185 (2000).

19. *Alifanov O. M.*, Inverse Heat Transfer Problems. Springer-Verl., Heidelberg (1994). (Int. Ser. Heat Mass Transfer. Springer Ser. Chem. Phys.)

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