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NUMERICALLY SOLVING THE IDENTIFICATION PROBLEM FOR THE LOWER COEFFICIENT OF A PARABOLIC EQUATION

P. N. Vabishchevich and V. I. Vasil′ev

Annotation. In the theory and practice of inverse problems for partial differential equations (PDEs) much attention is paid to the identification problem of coefficients from some additional information. This article deals with the problem of determining in a multidimensional parabolic equation the lower coefficient that depends only on time. To solve numerically a nonlinear inverse problem, linearized approximations in time are constructed using standard finite element procedures in space. The computational algorithm is based on a special decomposition, where the transition to a new time level is implemented via solving two standard elliptic problems. The numerical results presented here for a model 2D problem demonstrate capabilities of the proposed computational algorithms for approximate solving inverse problems.

Keywords: inverse problem, identification of the coefficient, parabolic partial differential equation, finite element approximation, difference scheme

The mathematical modeling of many applied problems of science and engineering results in the numerical solution of inverse problems [1–3]. Inverse problems often belong to the class of ill-posed (conditionally correct) problems, and so various regularization algorithms are employed to solve them numerically [4–6].

Particular attention should be given to inverse problems for PDEs [7, 8]. In this case, theoretical study includes the fundamental questions of uniqueness of a solution and its stability both from the viewpoint of the theory of differential equations [9, 10] and from the viewpoint of the theory of optimal control for distributed systems [11]. Many inverse problems are formulated as nonclassical problems for PDEs. To solve these problems approximately, emphasis is on the development of stable computational algorithms that take into account peculiarities of inverse problems [12, 13].

Among inverse problems for PDEs we distinguish the coefficient inverse problems that are associated with the identification of coefficients and/or the right-hand side of an equation using some additional information. When considering time-dependent problems, the identification of the coefficient dependences on space and time is usually separated into individual problems [8, 9]. In some cases, we have linear inverse problems (e.g., identification problems for the right-hand side of an equation); this situation essentially simplify their study.

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Much attention is paid to the problem of determining the lower coefficient of a second-order parabolic equation, where, in particular, the coefficient depends only on time. An additional condition is most often formulated as a specification of the solution at an interior point or as the average value integrated over the whole domain. The existence and uniqueness of the solution of such an inverse problem and the well-posedness of this problem in various function classes are examined, for instance, in [14–17].

Numerical methods for solving the identification problem of the lower coefficient of parabolic equations are considered in many articles [18–22]. In view of the practical use, we separately highlight the studies on numerical solution of inverse problems for multidimensional parabolic equations [23, 24]. To construct computational algorithms for the identification of the lower coefficient of a parabolic equation, the idea is widely used of transformation of the equation by introducing new unknowns, which results in a linear inverse problem.

In this paper, for a multidimensional parabolic equation we consider the problem of determining the lower coefficient that depends only on time. Approximation in space is performed by using standard finite elements [25, 26]. The main features of the nonlinear inverse problem are taken into account via a proper choice of the linearized approximation in time. Linear problems at a particular time level are solved on the basis of a special decomposition into two standard elliptic problems. The paper is organized as follows: In Section 1, considering a second-order parabolic equation we formulate the inverse identification problem of the lower coefficient. The computational algorithm basing on the linearization scheme is described in Section 2. Section 3 presents possibilities of the schemes with the second-order approximation in time. We describe the key aspects of theoretical justification in Section 4. Numerical results for a model 2D inverse problem are discussed in Section 5. Section 6 contains the results of calculations for data with noise.

1. Statement of the Problem

For simplicity, we restrict exposition to a 2D problem. Generalization to the 3D case is trivial. Let $\mathbf{x} = (x_1, x_2)$ and let Ω be a bounded polygon. The direct problem is formulated as follows. We search for $u(\mathbf{x}, t)$, $0 \le t \le T$, $T > 0$, such that it is a solution of the second-order parabolic equation

$$\frac{\partial u}{\partial t} - \operatorname{div}(k(\mathbf{x}) \operatorname{grad} u) + p(t)u = f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, \ 0 < t \le T. \tag{1}$$

The boundary and initial conditions are also specified:

$$k(\mathbf{x})\frac{\partial u}{\partial n} + g(\mathbf{x})u = 0, \quad \mathbf{x} \in \partial\Omega, \ 0 < t \le T, \tag{2}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{3}$$

where n is the normal to Ω. The formulation (1)–(3) presents the direct problem, where the right-hand side, and the coefficients of the equation as well as the boundary and initial conditions are specified.

Let us consider the inverse problem, where the coefficient $p(t)$ in (1) is unknown. The additional condition is often formulated as

$$\int_{\Omega} u(\mathbf{x}, t)\omega(\mathbf{x})\, d\mathbf{x} = \varphi(t), \quad 0 < t \le T, \tag{4}$$

where $\omega(\mathbf{x})$ is a weight function. In particular, choosing $\omega(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}^*)$, $\mathbf{x}^* \in \Omega$, where $\delta(\mathbf{x})$ is the Dirac δ-function, from (4) we get

$$u(\mathbf{x}^*, t) = \varphi(t), \quad 0 < t \le T. \tag{5}$$

We assume that the inverse problem of finding a pair of $u(\mathbf{x}, t)$, $p(t)$ from equations (1)–(3) and additional conditions (4) or (5) is well posed. The corresponding conditions for existence and uniqueness of a solution are available in the above-mentioned articles. In this paper, we consider only the numerical solution of these inverse problems, omitting the theoretical issues of the convergence of an approximate solution to the exact one.

From the nonlinear inverse problem we can proceed to the linear [17]. Suppose that

$$v(\mathbf{x}, t) = \chi(t) u(\mathbf{x}, t), \quad \chi(t) = \exp\left(\int_0^t p(\theta)\, d\theta\right).$$

Then from (1)–(3) we get

$$\frac{\partial v}{\partial t} - \operatorname{div}(k(\mathbf{x}) \operatorname{grad} v) = \chi(t) f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, \ 0 < t \le T,$$

$$k(\mathbf{x}) \frac{\partial v}{\partial n} + g(\mathbf{x}) v = 0, \quad \mathbf{x} \in \partial\Omega, \ 0 < t \le T,$$

$$v(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

The additional conditions (4) and (5), needed to identify uniquely $v(\mathbf{x}, t)$ and $\chi(t)$, take the form

$$\int_{\Omega} v(\mathbf{x}, t)\omega(\mathbf{x})\, d\mathbf{x} = \chi(t)\varphi(t), \quad 0 < t \le T,$$

$$u(\mathbf{x}^*, t) = \chi(t)\varphi(t), \quad 0 < t \le T.$$

The above transition from the nonlinear inverse problem to the linear problem is in common use for numerically solving identification problems (see, for example, [18, 21–23]). In this article, we focus on the original formulation of the inverse problem (1)–(4) (or (1)–(3), (5)) without the linear problem.

This allows us to consider more general problems in which the above passage transformation does not work. On the other hand, when pursuing our approach, the computational complexity of the algorithm for the original nonlinear inverse problem is not higher than for a linear inverse problem. In fact, for linear and nonlinear inverse problems, we use the same computational algorithm.

2. The Computational Algorithm

The inverse problem of determining the pair of $u(\mathbf{x},t)$, $p(t)$ is nonlinear. The standard approach is based on the simplest approximations in time and involves the iterative solution of the corresponding nonlinear problem for the evaluation of the approximate solution at a new level. In our work, we apply such approximations in time that lead to linear problems for evaluating the solution at the new time level.

Let us define the uniform grid in time

$$\overline{\omega}_\tau = \omega_\tau \cup \{T\} = \{t^n = n\tau,\ n = 0,1,\dots,N,\ \tau N = T\}$$

and put $y^n = y(t^n)$ and $t^n = n\tau$. Finite element approximations in space are employed. We perform a triangulation in Ω and introduce for this computational grid a finite-dimensional space $V \subset H^1(\Omega)$ of finite elements.

Using the fully implicit scheme for approximation in time, we obtain the variational problem:

$$\begin{gathered}
\int\limits_\Omega \frac{u^{n+1}-u^n}{\tau} v\,d\mathbf{x} + \int\limits_\Omega k(\mathbf{x}) \operatorname{grad} u^{n+1} \operatorname{grad} v\,d\mathbf{x} \\
+ \int\limits_{\partial\Omega} g(\mathbf{x}) u^{n+1} v\,dx + p^{n+1} \int\limits_\Omega u^{n+1} v\,d\mathbf{x} = \int\limits_\Omega f(\mathbf{x}, t^{n+1}) v\,d\mathbf{x}, \\
v \in V, \quad n = 0,1,\dots,N-1,
\end{gathered} \tag{6}$$

$$\int\limits_\Omega u^0 v\,d\mathbf{x} = \int\limits_\Omega u_0 v\,d\mathbf{x}. \tag{7}$$

The additional relations (4) and (5) take the form

$$\int\limits_\Omega u^{n+1} \omega(\mathbf{x})\,d\mathbf{x} = \varphi^{n+1}, \tag{8}$$

$$u^{n+1}(\mathbf{x}^*) = \varphi^{n+1}, \quad n = 0,1,\dots,N-1. \tag{9}$$

To evaluate the approximate solution $u^{n+1}(\mathbf{x})$, p^{n+1} at the new time level from (6)–(8) or (6), (7), (9), some iterative procedures are necessary. In solving time-dependent problems, the solution varies slightly when it passes from the previous time level to the next. This basic feature of time-dependent problems is widely used in numerically solving nonlinear problems by linearization procedures. We use a similar approach for the numerical solution of the inverse identification problem of the lower coefficient of a parabolic equation.

Instead of (6), we will solve the equation:

$$\begin{gathered}
\int\limits_\Omega \frac{u^{n+1}-u^n}{\tau} v\,d\mathbf{x} + \int\limits_\Omega k(\mathbf{x}) \operatorname{grad} u^{n+1} \operatorname{grad} v\,d\mathbf{x} \\
+ \int\limits_{\partial\Omega} g(\mathbf{x}) u^{n+1} v\,dx + p^{n+1} \int\limits_\Omega u^{n} v\,d\mathbf{x} = \int\limits_\Omega f(\mathbf{x}, t^{n+1}) v\,d\mathbf{x}, \\
v \in V, \quad n = 0,1,\dots,N-1.
\end{gathered} \tag{10}$$

In this case, the lower coefficient of the parabolic equation is taken at the upper time level, whereas the approximate solution $u(\mathbf{x}, t)$ is treated at the previous time level. Let us consider the solution procedure for (7), (9), (10) in detail.

For the approximate solution at the new time level u^{n+1}, we introduce the decomposition [13, 27]:

$$u^{n+1}(\mathbf{x}) = y^{n+1}(\mathbf{x}) + p^{n+1}w^{n+1}(\mathbf{x}). \tag{11}$$

To find $y^{n+1}(\mathbf{x})$, we employ the equation

$$\begin{aligned} &\int\limits_{\Omega} \frac{y^{n+1} - u^n}{\tau} v \, d\mathbf{x} + \int\limits_{\Omega} k(\mathbf{x}) \operatorname{grad} y^{n+1} \operatorname{grad} v \, d\mathbf{x} + \int\limits_{\partial\Omega} g(\mathbf{x}) y^{n+1} v \, dx \\ &\qquad = \int\limits_{\Omega} f(\mathbf{x}, t^{n+1}) v \, d\mathbf{x}, \quad v \in V, \ n = 0, 1, \dots, N-1. \end{aligned} \tag{12}$$

The function $w^{n+1}(\mathbf{x})$ is determined from

$$\begin{gathered} \int\limits_{\Omega} \frac{w^{n+1}}{\tau} v \, d\mathbf{x} + \int\limits_{\Omega} k(\mathbf{x}) \operatorname{grad} w^{n+1} \operatorname{grad} v \, d\mathbf{x} + \int\limits_{\partial\Omega} g(\mathbf{x}) w^{n+1} v \, dx = -\int\limits_{\Omega} u^n v \, d\mathbf{x}, \\ v \in V, \ n = 0, 1, \dots, N-1. \end{gathered} \tag{13}$$

In view of (11)–(12), equation (10) holds automatically for all p^{n+1}.

To evaluate p^{n+1}, we apply (9) (or (8)). The insertion of (11) into (9) yields

$$p^{n+1} = \frac{1}{w^{n+1}(\mathbf{x}^*)} (\varphi^{n+1} - y^{n+1}(\mathbf{x}^*)). \tag{14}$$

The fundamental point of applicability of this algorithm is associated with the condition $w^{n+1}(\mathbf{x}^*) \neq 0$. The auxiliary function $w^{n+1}(\mathbf{x})$ is determined from the grid elliptic equation (13). The property of having a fixed sign for $w^{n+1}(\mathbf{x})$ is followed, in particular, from the same property of the solution at the previous time level $u^n(\mathbf{x})$. Such constraints on the solution can be provided by the corresponding restrictions on the input data of the inverse problem. In any case, this problem requires special and careful consideration. We assume that the constraint $w^{n+1}(\mathbf{x}^*) \neq 0$ is satisfied. Some evidential arguments in favor of this are given in Section 4.

In solving (8)–(10), instead of (14), we have

$$p^{n+1} = \frac{1}{\int\limits_{\Omega} w^{n+1} \omega(\mathbf{x}) \, d\mathbf{x}} \left(\varphi^{n+1} - \int\limits_{\Omega} y^{n+1} \omega(\mathbf{x}) \, d\mathbf{x} \right) \tag{15}$$

under the condition that

$$\int\limits_{\Omega} w^{n+1} \omega(\mathbf{x}) \, d\mathbf{x} \neq 0.$$

In this case, the additional restrictions are formulated on $\omega(\mathbf{x})$, e.g., its fixed sign in Ω.

Thus, the computational algorithm for solving the inverse problem (1)–(4) (or (1)–(3), (5)) basing on the linearized scheme (7), (8), (10) (or (7), (9), (10)) involves the solution of two standard grid elliptic equations for the auxiliary functions $y^{n+1}(\mathbf{x})$ (equation (12)) and $w^{n+1}(\mathbf{x})$ (equation (13)), the further evaluation of p^{n+1} from (15) (or (14)), and the final calculation $u^{n+1}(\mathbf{x})$ from (11).

3. Scheme of the Second-Order Accuracy

The nonlinear inverse problem (1)–(4) is characterized by a quadratic nonlinearity. When using the scheme with linearization (7), (8), (10), the nonlinear term is approximated with the first order with respect to τ. It is possible to apply the linearized scheme of the second order. Let us consider the approximation

$$a(t^{n+1/2})b(t^{n+1/2}) = \frac{1}{2}a(t^{n+1})b(t^n) + \frac{1}{2}a(t^n)b(t^{n+1}) + O(\tau^2).$$

The approximation of (1) with the boundary conditions (2) using the Crank–Nicolson scheme yields the linearized scheme

$$\begin{aligned}&\int\limits_\Omega \frac{u^{n+1}-u^n}{\tau} v\, d\mathbf{x} + \frac{1}{2}\int\limits_\Omega k(\mathbf{x}) \operatorname{grad} u^{n+1} \operatorname{grad} v d\mathbf{x} + \frac{1}{2}\int\limits_{\partial\Omega} g(\mathbf{x}) u^{n+1} v\, dx \\ &+\frac{1}{2}\int\limits_\Omega k(\mathbf{x}) \operatorname{grad} u^n \operatorname{grad} v d\mathbf{x} + \frac{1}{2}\int\limits_{\partial\Omega} g(\mathbf{x}) u^n v\, dx + \frac{1}{2}p^{n+1}\int\limits_\Omega u^n v\, d\mathbf{x} + \frac{1}{2}p^n \int\limits_\Omega u^{n+1} v\, d\mathbf{x} \\ &= \int\limits_\Omega f(\mathbf{x}, t^{n+1/2}) v\, d\mathbf{x}, \quad v \in V, \ n = 0, 1, \ldots, N-1. \end{aligned} \tag{16}$$

The scheme (7), (8), (16) belongs to the class of linearized schemes. In comparison with (7), (8), (10), it has a higher order of accuracy in time.

To implement (7), (8), (16), we again use (11). In this case, for $y^{n+1}(\mathbf{x})$, we have

$$\begin{aligned}&\int\limits_\Omega \frac{y^{n+1}-u^n}{\tau} v\, d\mathbf{x} + \frac{1}{2}\int\limits_\Omega k(\mathbf{x}) \operatorname{grad} y^{n+1} \operatorname{grad} v\, d\mathbf{x} + \frac{1}{2}\int\limits_{\partial\Omega} g(\mathbf{x}) y^{n+1} v\, dx \\ &+\frac{1}{2}\int\limits_\Omega k(\mathbf{x}) \operatorname{grad} u^n \operatorname{grad} v d\mathbf{x} + \frac{1}{2}\int\limits_{\partial\Omega} g(\mathbf{x}) u^n v\, dx + \frac{1}{2}p^n \int\limits_\Omega y^{n+1} v\, d\mathbf{x} \\ &= \int\limits_\Omega f(\mathbf{x}, t^{n+1/2}) v\, d\mathbf{x}, \quad v \in V, \ n = 0, 1, \ldots, N-1. \end{aligned} \tag{17}$$

The auxiliary function $w^{n+1}(\mathbf{x})$ is defined as the solution of the equation

$$\begin{aligned}&\int\limits_\Omega \frac{w^{n+1}}{\tau} v\, d\mathbf{x} + \frac{1}{2}\int\limits_\Omega k(\mathbf{x}) \operatorname{grad} w^{n+1} \operatorname{grad} v d\mathbf{x} + \frac{1}{2}\int\limits_{\partial\Omega} g(\mathbf{x}) w^{n+1} v\, dx \\ &+\frac{1}{2}p^n \int\limits_\Omega w^{n+1} v\, d\mathbf{x} = -\frac{1}{2}\int\limits_\Omega u^n v\, d\mathbf{x}, \quad v \in V, \ n = 0, 1, \ldots, N-1. \end{aligned} \tag{18}$$

Further, we employ (15) as in the case of the first-order scheme.

The Crank–Nicolson scheme for numerically solving the direct problems for parabolic equations is not very often used in computational practice. It is inferior to the fully implicit scheme in regard to conservation of monotonicity (fulfilment of the maximum principle for the grid problem), it has poor asymptotic properties for solving problems with large integration time, and it is not an unconditionally SM-stable scheme [28, 29]. For this reason, it is appropriate to consider another variant of linearization of this inverse problem, where the second-order approximation is applied only for the nonlinear term. In this case, instead of (10), (16) we put

$$\int_{\Omega} \frac{u^{n+1} - u^n}{\tau} v \, d\mathbf{x} + \int_{\Omega} k(\mathbf{x}) \operatorname{grad} u^{n+1} \operatorname{grad} v \, d\mathbf{x} + \int_{\partial\Omega} g(\mathbf{x}) u^{n+1} v \, dx$$
$$+ \frac{1}{2} p^{n+1} \int_{\Omega} u^n v \, d\mathbf{x} + \frac{1}{2} p^n \int_{\Omega} u^{n+1} v \, d\mathbf{x} = \int_{\Omega} f(\mathbf{x}, t^{n+1/2}) v \, d\mathbf{x},$$
$$v \in V, \quad n = 0, 1, \ldots, N-1. \tag{19}$$

The numerical implementation of (7), (8), (19) is performed in the standard way using (11).

4. Key Aspects of Theoretical Justification

To validate the above computational algorithm, we need a thorough inspection. This involves the following issues: solvability of the problem at the discrete level, correctness of the computational algorithm and convergence of the approximate solution to the exact. Such a detailed examination is the subject of another publication. Here we restrict exposition to discussion of some key aspects of correctness of the algorithm for solving the inverse identification problem of the lower coefficient for a parabolic equation.

Let us rewrite the boundary value problem in operator form. On the set of functions satisfying (2) under the constraint $g(\mathbf{x}) > 0$, $\mathbf{x} \in \partial\Omega$, we consider the Cauchy problem for the equation

$$\frac{\partial u}{\partial t} + Lu + p(t)u = f, \quad 0 < t \le T, \tag{20}$$

where

$$Lu = -\operatorname{div}(k(\mathbf{x}) \operatorname{grad} u).$$

To guarantee the correctness of the computational algorithm (11)–(14) for solving (1)–(3, (5), we must ensure that $w^{n+1}(\mathbf{x}^*) \neq 0$. In this study, it is natural to focus on the use of the maximum principle for parabolic equations, and after discretization in time we do the same for elliptic equations [30–32].

When considering the boundary value problem of diffusion-reaction (20), (3), the approximation in time for this equation is carried out depending on the sign of $p(t)$ [33, 34]. In particular, if $p(t) \geq 0$, then the unconditionally stable schemes are constructed using the implicit approximation of the lower coefficient; i.e. (see (6)),

$$\frac{u^{n+1} - u^n}{\tau} + Lu^{n+1} + p^{n+1} u^{n+1} = f^{n+1}.$$

For $p(t) \leq 0$, we need to employ the explicit approximation (see (11)):

$$\frac{u^{n+1} - u^n}{\tau} + Lu^{n+1} + p^{n+1} u^n = f^{n+1}, \quad n = 0, 1, \ldots, N-1. \tag{21}$$

The initial condition (3) yields

$$u^0 = u_0. \tag{22}$$

For the coefficient $p(t)$ with the alternating sign, the unconditionally stable schemes are constructed via the explicit-implicit approximations for the lower term. If we use (21) for the problem (20), (3) with $p(t) \geq 0$, then the stability of the scheme (21), (22) takes place under rather weak restrictions on the time meshsize [34].

Theorem 1. *Assume that* $u_0 > 0$, $f^{n+1} > 0$, $n = 0, 1, \dots, N-1$ *in the scheme* (21), (22). *Then*

$$u^{n+1} > 0, \quad n = 0, 1, \dots, N-1, \tag{23}$$

for all τ, *if* $p^{n+1} < 0$, $n = 0, 1, \dots, N-1$, *and this holds for*

$$\tau \le \frac{1}{p_m}, \tag{24}$$

if $p_m \ge p^{n+1} > 0$, $n = 0, 1, \dots, N-1$.

Proof. Rewrite (21) as

$$\frac{1}{\tau} u^{n+1} + L u^{n+1} = f^{n+1} + u^n \left(\frac{1}{\tau} - p^{n+1} \right), \quad n = 0, 1, \dots, N-1.$$

The right-hand side is always positive for $p^{n+1} < 0$, $n = 0, 1, \dots, N-1$; this is true for $p^{n+1} > 0$, $n = 0, 1, \dots, N-1$ under the restriction on the time meshsize (24). On the basis of the maximum principle for elliptic equations [32, 35], if the right-hand side is positive, then we have a positive solution, i.e. the desired inequality (23).

The numerical algorithm for solving the inverse problem (1)–(4) (or (1)–(3), (5)) is based on (11). In this case, from (21) we obtain the equation for y^{n+1}:

$$\frac{y^{n+1} - u^n}{\tau} + L y^{n+1} = f^{n+1}, \quad n = 0, 1, \dots, N-1. \tag{25}$$

For w^{n+1}, we have

$$\frac{1}{\tau} w^{n+1} + L w^{n+1} = -u^n, \quad n = 0, 1, \dots, N-1. \tag{26}$$

Let us consider in detail the case where for the direct problem we have $p^{n+1} < 0$, $n = 0, 1, \dots, N-1$, and the coefficient p^{n+1} is determined from (14).

Theorem 2. *If* $u_0 > 0$, $f^{n+1} > 0$, $n = 0, 1, \dots, N-1$, *and* $p^{n+1} < 0$, $n = 0, 1, \dots, N-1$, *then the solution of* (9), (21), (22) *may be represented in the form* (11), (14), (25), (26).

Proof. Correctness of the computational algorithm follows from the condition $w^{n+1}(\mathbf{x}^*) \ne 0$. In view of (26), it is sufficient to show the fulfilment of (23). Prove this by induction. We have $u^0 > 0$ and let $u^n > 0$. From (26) we get $w^{n+1} < 0$. From (25), we have also $y^{n+1} > 0$.

For $u^{n+1} - y^{n+1}$, from (21) and (25), we get

$$\frac{u^{n+1} - y^{n+1}}{\tau} + L(u^{n+1} - y^{n+1}) = -p^{n+1} u^n.$$

By the above, $u^{n+1} - y^{n+1} > 0$ and so $u^{n+1} > 0$. Note that $\varphi^{n+1} - y^{n+1}(\mathbf{x}^*) > 0$, and in the determination of the coefficient according to (14), we have $p^{n+1} < 0$. Thus, we remain in the class of negative coefficients.

We have considered the variant with negative coefficients ($p^{n+1} < 0$, $n = 0, 1, \dots, N-1$) and additional measurements at one point (the condition (14)). Other variants can be considered in a similar way. It should be noted that for positive coefficients ($p^{n+1} > 0$, $n = 0, 1, \dots, N-1$), there does exist restriction on the time meshsize (21); to determine the coefficient according to (15), is seems natural to require the fixed sign for the function $\omega(\mathbf{x})$.

5. Numerical Examples

To demonstrate possibilities of the above linearization schemes for solving the identification problem of the lower coefficient of a parabolic equation, we consider a 2D model problem. In the examples below, we put

$$k(\mathbf{x}) = 1, \quad f(\mathbf{x}, t) = 0, \quad u_0(\mathbf{x}) = 1, \quad \mathbf{x} \in \Omega, \quad g(\mathbf{x}) = 10, \quad \mathbf{x} \in \partial\Omega.$$

The problem is considered on a triangular grid, which consists of 1.180 nodes (2.230 triangles) and is shown in Fig. 1. Here Ω is the trapezoid with the vertices coordinates $(0,0), (0,1), (1.5, 0.5), (1.5, 0)$. The calculations were carried out for $T = 0.1$. The coefficient $p(t)$ is taken in the form

$$p(t) = \begin{cases} 1000t, & 0 < t \le \frac{1}{2}T, \\ 0, & \frac{1}{2}T < t \le T. \end{cases} \tag{27}$$

The solution of (1)–(3) at the observation point is depicted in Fig. 2. It was obtained using the fully implicit scheme with different time meshsizes. The solution at the final time moment is presented in Fig. 3.

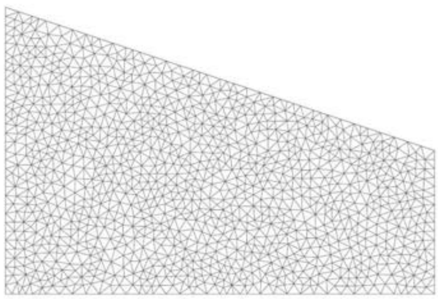

Fig. 1. Computational grid

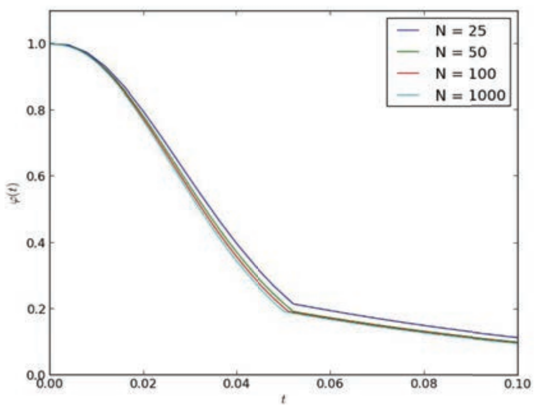

Fig. 2. Solution of the direct problem at the observation point

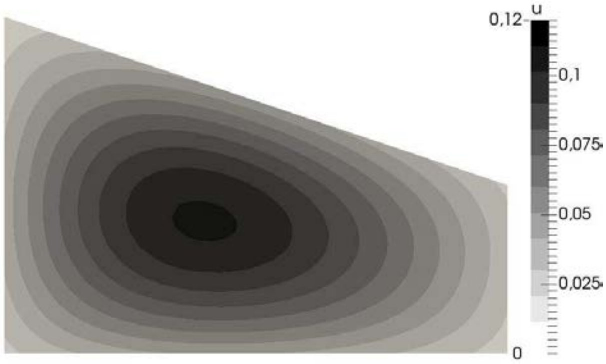

Fig. 3. Solution of the direct problem at $t = T$

The results of solving the inverse problem with various grids in time are shown in Fig. 4. In the study of the influence of parameters of computational algorithm we need to use the same input data. In our case, the input data used by the numerical solution of the direct problem for a very detailed grid over time. The solution of the direct problem obtained with $N = 1000$ is used as the input data (the function $\varphi(t)$ in the condition (5)). It is easy to see that the approximate solution of the inverse problem converges with decreasing the time meshsize. These results were obtained using the first-order scheme (7), (9), (10).

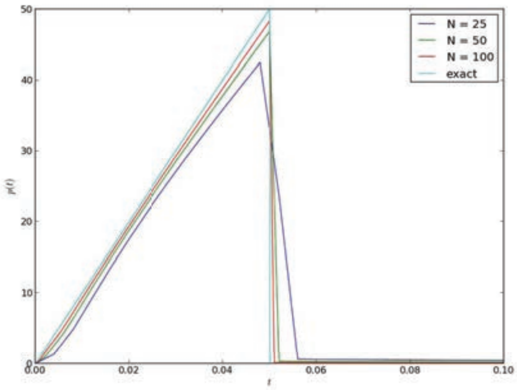

Fig. 4. The solution of the inverse problem

The numerical results for the above problem using the second-order scheme (7), (9), (16) are shown in Fig. 5. For the discontinuous right-hand side (20), we observe some characteristic wiggles of the identified coefficient. These oscillations of the approximate solution are typical for the scheme (7), (9), (19).

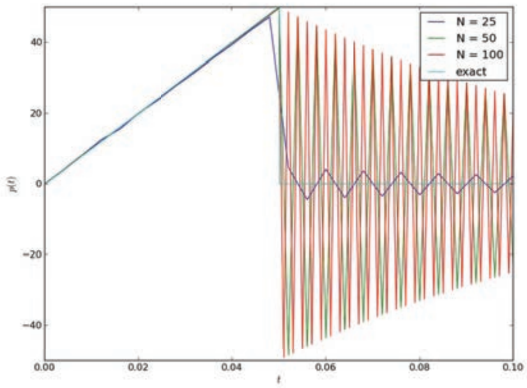

Fig. 5. The solution of the inverse problem (the second-order scheme)

If the desired solution (the coefficient $p(t)$) is smooth, then the effect of using the second-order approximation is clearly expressed. As an example, we present the results of numerically solving the inverse problem, where the lower coefficient (the exact solution) has the form

$$p(t) = \frac{1000t}{1 + 500t^2}.$$

The approximate solution obtained via the first-order scheme (7), (9), (10) is shown in Fig. 6, whereas Fig. 7 demonstrates the computations by the second-order scheme (7), (9), (16).

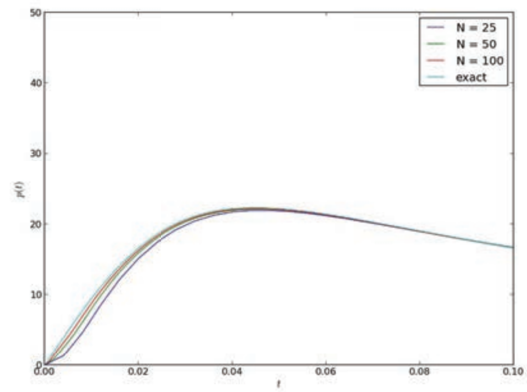

Fig. 6. Solution of the inverse problem (scheme of first order; smooth solution)

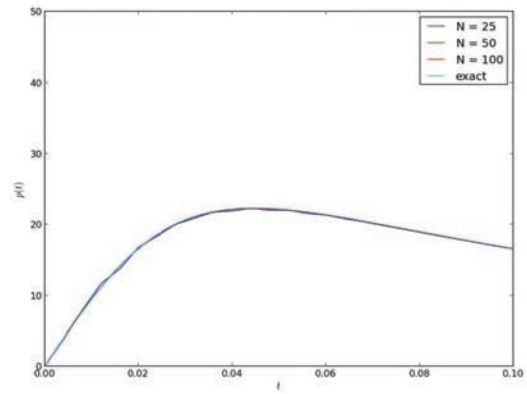

Fig. 7. Solution of the inverse problem (scheme of the second order; smooth solution)

6. Data with a Random Noise

In the development of numerical algorithms for the approximate solution of inverse problems, much attention is paid to the case where the input data are given with an error. In our identification problem of the lower coefficient in a parabolic equation using observations of the solution at an interior point (the problem (1)–(3), (5)), the fundamental problem is to study the influence of inaccuracies in specifying the function $\varphi(t)$, $0 < t \leq T$ on the accuracy of the identification of $p(t)$, $0 < t \leq T$ in equation (1).

Before considering the problems with noisy input data, we have to specify the class of these input data. In our case, $\varphi(t)$, $0 < t \leq T$, is the solution of the boundary value problem (1)–(3) for a given coefficient $p(t)$, $0 < t \leq T$. By the above, for the classical solution of this boundary value problem, $\varphi(t)$, $0 < t \leq T$, and the first derivative of $\varphi(t)$ must be continuous ($\varphi(t) \in C^1(0,T)$). This boundary value problem is solved numerically, and for the convergence of the approximate solution of this problem to the exact one, we have to assume additional smoothness of the solution, for example, $\varphi(t) \in C^2(0,T)$.

In our model problem, the input data are given using the numerical solution of the direct problem which is derived on the fine grid in time ($N = 1000$). To illustrate the work with noisy input data, we introduce an error in the values of the function $\varphi(t)$ at the nodes of the time grid. We set

$$\varphi_\delta(t^n) = \varphi(t^n) + \delta(2\sigma - 1), \quad t^n = n\tau, \quad n = 1, 2, \ldots, N, \tag{28}$$

where σ is a random variable uniformly distributed on the interval from 0 to 1, and δ determines an error level of the input data specification. Fig. 8 shows the noisy data for solving the problem (1)–(3) with the right-hand side (20) and $\delta = 0.01$.

In view of the above requirements on sufficiently smooth input data, the grid function $\varphi_\delta(t^n)$, $n = 1, 2, \ldots, N$, can be smoothed. When considering time-dependent problems, a smoothing procedure is conducted under the condition that the observed function $\varphi(t)$ is given exactly at $t = 0$, when it is determined by the initial

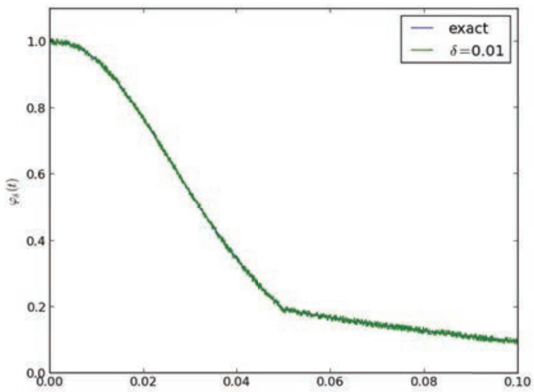

Fig. 8. Data with errors

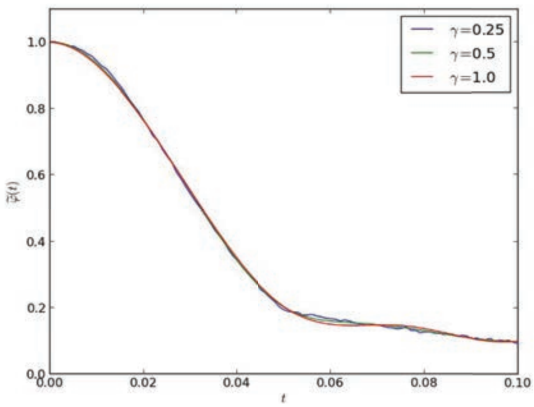

Fig. 9. Smoothed function $\widetilde{\varphi}(t)$ at different values of γ

state. For smoothing the one-dimensional function $\varphi_\delta(t^n)$, $n = 1, 2, \dots, N$, we use smoothing B-splines [36].

In the class $\varphi(t) \in C^2(0, T)$, we can apply cubic smoothing splines, where the approximating function $\widetilde{\varphi}(t)$ is determined from the condition of minimum

$$\min \int_0^T (\widetilde{\varphi}''(t))^2 \, dt : \sum_{n=1}^N (\widetilde{\varphi}(t^n) - \varphi_\delta(t^n))^2 = \gamma N \delta^2, \quad \widetilde{\varphi}(0) = \varphi(0).$$

The influence of the numerical parameter γ that is used in the error level specification is illustrated in Fig. 9–11 for the approximating function $\widetilde{\varphi}(t)$ and its first and second derivatives. The result of solving the inverse problem of the coefficient identification is presented in Fig. 12 for various values of γ, $N = 100$.

On the basis of these calculations we use $\gamma = 0.5$ when processing the input data. The reconstruction of the lower coefficient of the parabolic equation is shown in Fig. 13 for different levels of error in the input data ($N = 100$). For the problem

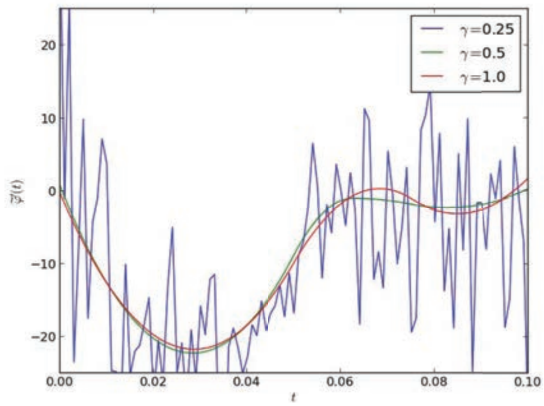

Fig. 10. The first derivative of $\widetilde{\varphi}(t)$ at different values of γ

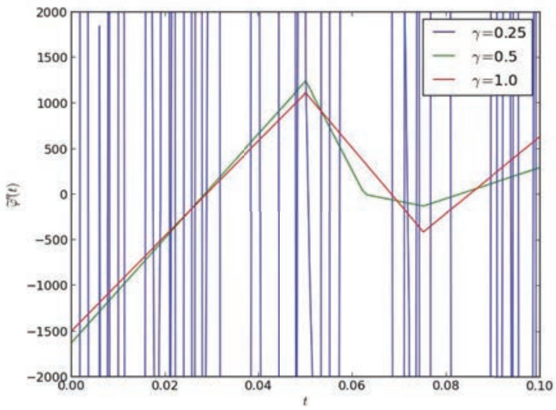

Fig. 11. The second derivative of $\widetilde{\varphi}(t)$ at different values of γ

with a smooth coefficient, the convergence of the solution of the inverse problem is depicted in Fig. 14 for a decreasing error level in the input data.

7. Conclusions

1. A computational algorithm was proposed for numerically solving the inverse identification problem of the lower coefficient in a parabolic equation using observations of the solution at an interior point of a computational domain or the average value integrated over the whole domain. The algorithm is based on implicit approximations in time with solving linear problems at every time level.

2. The numerical implementation is based on a decomposition of the approximate solution, where the transition to a new time level involves the solution of two standard grid elliptic problems.

3. Capabilities of the computational algorithm were illustrated by solving the model two-dimensional parabolic problem. The algorithm demonstrates high robust-

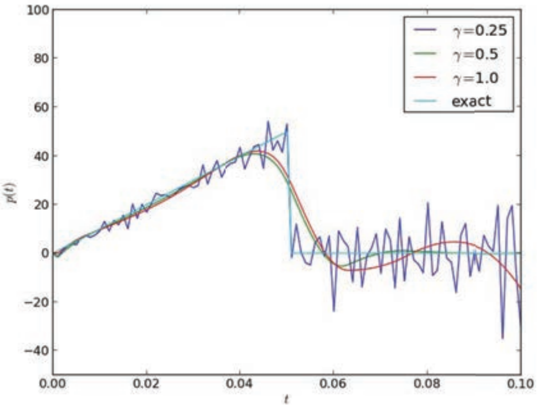

Fig. 12. Coefficient reconstruction

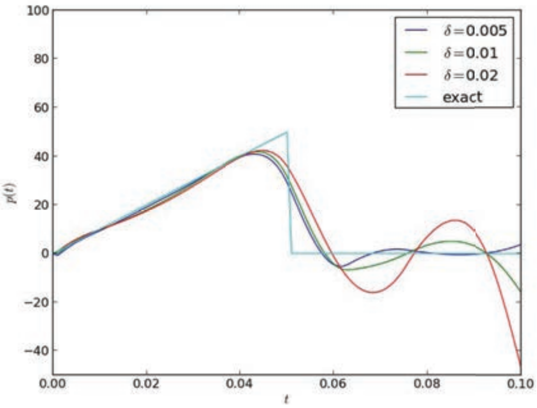

Fig. 13. Solution of the inverse problem at different error levels

ness in the case, where an unknown coefficient is taken from the previous time level.

4. If input data are specified with errors, then preliminary smoothing by means of B-splines is carried out taking into account the error level. Numerical solutions of the model problem demonstrate the convergence of the approximate solution of the inverse problem to the exact one if the error level of input data is reduced.

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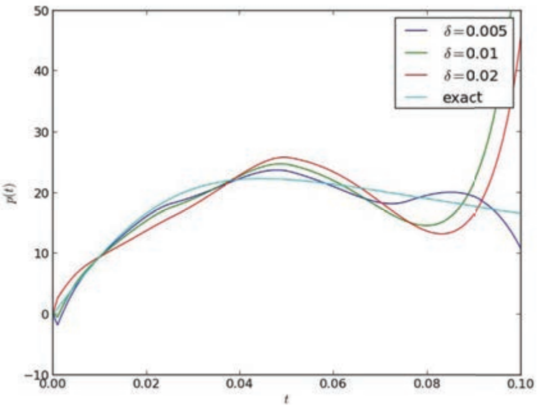

Fig. 14. Reconstruction of the smooth coefficient

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P. N. Vabishchevich
Nuclear Safety Institute, Russian Academy of Sciences
115191, Moscow, Russia
vabishchevich@gmail.com

V. I. Vasil′ev
North-Eastern Federal University,
677000, Yakutsk, Russia
vasvasil@mail.ru