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Simultaneously Identify the Leading Coefficient and Right-hand Side in a Parabolic Equation

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Simultaneously Identify the Leading Coefficient and Right-hand Side in a Parabolic Equation

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Abstract. In this article, we propose a new method for the numerical solution of a kind of coefficient inverse problem, in which the time dependent leading coefficient and the right hand side in a parabolic equation need to be identified simultaneously, based on additional information of the solutions at interior points in the computational domain. To solve the nonlinear problem, we use linearized second-order scheme for linearizing the quadratic nonlinear term in time and finite element approximation is used in the space. Based on a special decomposition, at new time level, the original problem is transformed into three standard elliptic problems. The results of numerical experiments are presented to confirm the capability and efficiency of the proposed computational algorithm.

1. Introduction

An inverse problem is a type of problem corresponding to the direct one, in an inverse problem, the unknown causes need to be found from the measured consequences with some additional conditions. Mathematical modeling of many application problems involves inverse problems [1, 2], a lot of important characteristics of the media can be described by the solution of an inverse problem, such as conductivity, elasticity parameters [3, 4], wave density and velocity etc., therefore, to find the solution of inverse problem becomes crucial.

According to J. Hadamard [5], the inverse problems for partial differential equation usually belong to nonclassical problems, they are posed as the class of ill-posed or conditional well-posed problems. Since the minimal change of the input value will bring about a huge change in the results, it is difficult for the traditional numerical methods to obtain a more accurate solution.Therefore, it is becoming more and more important to find a high-precision and stable algorithm for the numerical solution of inverse problem. Many works have been done in the study of inverse problem [6–8].

The problem we study in this paper is subject to the inverse problem of coefficient. In a coefficient inverse problem, we need to get the unknown coefficient and the solution of the master equation by using some additional conditions, these conditions are most often stated as specifications of solution values at some internal points [9]. Most of the coefficient inverse problems are nonlinear, except for only solving the right-hand side in the problem, which makes the calculation more complicated. The work to find the right-hand side and the leading coefficient simultaneously is a nonlinear problem.

The coefficient inverse problems attracted many researchers' attention. [10–12] gave the research of existence and uniqueness of the solution of the problem to identify lowest coefficient and well-posedness of this problem in various functional classes. Numerical solution of the lower coefficient in parabolic equation was considered in many works[13–15]. And also very concerned about finding the

leading coefficient in a parabolic equation, the well-posedness in various functional classes and theoretical studies about existence and uniqueness of the solution were studied in [16–18], numerical methods for finding the leading coefficient numerically were also considered in many works [9, 19–21].
In this paper, we concentrate on finding numerical solutions of time dependent right-hand side and the leading coefficient in a parabolic equation, simultaneously. To solve this problem, for the approximation in space, we use the standard finite element method and at new time level, on the basis of a special decomposition the problem is transformed into three standard elliptic problems. In dealing with quadratic nonlinear term, at the new time step, we construct linearized second-order scheme for getting the approximate solution from a linear problem.

2. The Problem Formulation

Considering the following parabolic problem,

$$u_t - a(t)\mathrm{div}\left(k(\boldsymbol{x})\mathrm{grad}\, u\right) + c(\boldsymbol{x})u = b(t)f(\boldsymbol{x},t), \qquad \boldsymbol{x}\in\Omega, \quad 0<t\le T \tag{1}$$

where $k(\boldsymbol{x})>0$, $a(t)>0$, $c(\boldsymbol{x})>0$ and Ω is a bounded domain, the initial condition,

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \quad x\in\Omega, \quad b(0)=b_0, \tag{2}$$

and the boundary condition,

$$a(t)k(\boldsymbol{x})\frac{\partial u}{\partial n} + g(\boldsymbol{x})u = 0, \quad \boldsymbol{x}\in\partial\Omega, \quad 0<\mathrm{t}\le\mathrm{T} \tag{3}$$

where n is the normal of the boundary $\partial\Omega$ and $g(\boldsymbol{x})>0$.
In this article, our research focuses on simultaneously solving numerical solutions of leading coefficientn $a(t)$ and the right-hand side $b(t)$ and ignoring relevant theoretical research. As we mentioned above, solving a coefficient inverse problem usually need additional conditions. The additional conditions can be obtained from the values of solution at some interior points, which can be given as follows,

$$\begin{aligned} u(\boldsymbol{x}^*,t) &= \Phi(t), \quad 0<t\le T, \\ u(\boldsymbol{x}^{**},t) &= \Psi(t), \quad 0<t\le T, \end{aligned} \tag{4}$$

where $\boldsymbol{x}^*$ and $\boldsymbol{x}^{**}$ are two points in Ω, $\Phi(t)$ and $\Psi(t)$ are known functions.
The inverse problem in this paper is to find the time dependent leading coefficient $a(t)$ and the right-hand side function $b(t)$ simultaneously from (1) – (4), to linearize the nonlinear problem we use the linearized second-order scheme [8, 9, 22].

3. The Computational Algorithm

Let's consider inverse problem (1) - (4), for the numerical solution of this inverse problem, we define a uniform grid in time,

$$\omega_\tau = \left\{t^n \mid t^n = n\tau, \ \ n=0,1,\cdots,N, \ N\tau = T\right\}.$$

For the linearization of the quadratic nonlinear term, we will use the linearized scheme which is based on approximation of $\alpha(t)\beta(t)$,

$$\alpha\left(t^{n+\frac{1}{2}}\right)\beta\left(t^{n+\frac{1}{2}}\right) = \frac{1}{2}\alpha\left(t^{n+1}\right)\beta\left(t^n\right) + \frac{1}{2}\alpha\left(t^n\right)\beta\left(t^{n+1}\right) + o\left(\tau^2\right).$$

Based on the Crank-Nicolson scheme and combine with the standard finite element method, we get the linearized variational problem as follows,

$$\int_\Omega \frac{u^{n+1}-u^n}{\tau} v\mathrm{d}\boldsymbol{x} + \frac{a^n}{2}\int_\Omega k(\boldsymbol{x})\mathrm{grad}\, u^{n+1}\mathrm{grad}\, v\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\partial\Omega} g(\boldsymbol{x})u^{n+1}v\mathrm{d}s + \frac{a^{n+1}}{2}\int_\Omega k(\boldsymbol{x})\mathrm{grad}\, u^n\mathrm{grad}\, v\mathrm{d}\boldsymbol{x}$$
$$+\frac{1}{2}\int_{\partial\Omega} g(\boldsymbol{x})u^n v\mathrm{d}s + \frac{1}{2}\int_\Omega c(\boldsymbol{x})u^n v\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_\Omega c(\boldsymbol{x})u^{n+1}v\mathrm{d}\boldsymbol{x} = \frac{1}{2}\int_\Omega b^n f^{n+1}v\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_\Omega b^{n+1}f^n v\mathrm{d}\boldsymbol{x}, \quad (5)$$

$$\int_\Omega u(\boldsymbol{x},0) = \int_\Omega u_0(\boldsymbol{x})v\mathrm{d}\boldsymbol{x}, \quad (6)$$

where $n = 0, 1, \dots, N - 1$ and $v \in V$.
We use the new decomposition for the solution $u^{n+1}(\boldsymbol{x})$, the similar decomposition was used in our previous works [23, 24] for simultaneously determine the time dependent lowest coefficient and right hand side function in parabolic equation,

$$u^{n+1}(\boldsymbol{x}) = v^{n+1}(\boldsymbol{x}) + a^{n+1}w_1^{n+1}(\boldsymbol{x}) + b^{n+1}w_2^{n+1}(\boldsymbol{x}). \quad (7)$$

To find $v^{n+1}(\boldsymbol{x})$, substituted (7) in (5), we employ the equation,

$$\int_\Omega \frac{y^{n+1}-u^n}{\tau} v\mathrm{d}\boldsymbol{x} + \frac{a^n}{2}\int_\Omega k(\boldsymbol{x})\mathrm{grad}\, v\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\partial\Omega} g(\boldsymbol{x})y^{n+1}v\mathrm{d}s + \frac{1}{2}\int_\Omega c(\boldsymbol{x})y^{n+1}v\mathrm{d}\boldsymbol{x}$$
$$+\frac{1}{2}\int_{\partial\Omega} g(\boldsymbol{x})u^n v\mathrm{d}s + \frac{1}{2}\int_\Omega c(\boldsymbol{x})u^n v\mathrm{d}\boldsymbol{x} = \frac{1}{2}\int_\Omega b^n f^{n+1}v\mathrm{d}\boldsymbol{x}, \quad (8)$$

The functions $w_1^{n+1}(\boldsymbol{x})$ and $w_2^{n+1}(\boldsymbol{x})$ are determined from,

$$\int_\Omega \frac{w_1^{n+1}}{\tau} v\mathrm{d}\boldsymbol{x} + \frac{a^n}{2}\int_\Omega k(\boldsymbol{x})\mathrm{grad}\, w_1^{n+1}\mathrm{grad}\, v\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\partial\Omega} g(\boldsymbol{x})w_1^{n+1}v\mathrm{d}s$$
$$+\frac{1}{2}\int_\Omega c(\boldsymbol{x})w_1^{n+1}v\mathrm{d}\boldsymbol{x} = -\frac{1}{2}\int_\Omega k(\boldsymbol{x})\mathrm{grad}\, u^n\mathrm{grad}\, v\mathrm{d}\boldsymbol{x}, \quad (9)$$

$$\int_\Omega \frac{w_2^{n+1}}{\tau} v\mathrm{d}\boldsymbol{x} + \frac{a^n}{2}\int_\Omega k(\boldsymbol{x})\mathrm{grad}\, w_2^{n+1}\mathrm{grad}\, v\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\partial\Omega} g(\boldsymbol{x})w_2^{n+1}v\mathrm{d}s$$
$$+\frac{1}{2}\int_\Omega c(\boldsymbol{x})w_2^{n+1}v\mathrm{d}\boldsymbol{x} = \frac{1}{2}\int_\Omega f^n v\mathrm{d}\boldsymbol{x}. \quad (10)$$

To evaluate a^{n+1} and b^{n+1}, the addition conditions (4) are used, substituted (7) into (4), we get

$$\begin{aligned} a^{n+1}w_1^{n+1}(\boldsymbol{x}^*) + b^{n+1}w_2^{n+1}(\boldsymbol{x}^*) &= \Phi^{n+1} - v^{n+1}(\boldsymbol{x}^*), \\ a^{n+1}w_1^{n+1}(\boldsymbol{x}^{**}) + b^{n+1}w_2^{n+1}(\boldsymbol{x}^{**}) &= \Psi^{n+1} - v^{n+1}(\boldsymbol{x}^{**}), \end{aligned} \quad (11)$$

we assume $w_1^{n+1}(\boldsymbol{x}^*)w_2^{n+1}(\boldsymbol{x}^{**}) - w_1^{n+1}(\boldsymbol{x}^{**})w_2^{n+1}(\boldsymbol{x}^*) \neq 0$, to ensure the solvability of linear system (11), the auxiliary functions v^{n+1}, w_1^{n+1} and w_2^{n+1} are determined from (8) – (10), the evaluation of a^{n+1} and b^{n+1} from (11), and $u^{n+1}(\boldsymbol{x})$ can be get from the relation (7).

4. Numerical Example

To confirm the capability of the numerical algorithm in the previous section for solving the coefficient inverse problems in parabolic equation, we take a two-dimensional numerical example and for high-dimensional situations, it can be directly extended. To evaluate the numerical errors, we adopted the two errors $\epsilon_1(t) = \left|\widetilde{a}(t) - a(t)\right|$, $\epsilon_2(t) = \left|\widetilde{b}(t) - b(t)\right|$, where $\widetilde{a}(t), \widetilde{b}(t)$ are the numerical solutions of a and b at time t, $a(t)$ and $b(t)$ are the exact ones.

***Example**:* In this example, the computational domain $\Omega = [0, 1]\times[0, 1]$ with conditions,

$$k(\boldsymbol{x}) = 1, \quad g(\boldsymbol{x}) = 0, \quad c(\boldsymbol{x}) = 0$$
$$f(\boldsymbol{x},t) = 10\pi^2\cos(\pi x_1)\cos(2\pi x_2), \quad \boldsymbol{x} = (x_1, x_2) \in \Omega$$

and the initial conditions are $u_0(\mathbf{x}) = \cos(\pi x_1)\cos(2\pi x_2)$, $b_0 = 1$. The leading coefficient $a(t)$ and right-hand side $b(t)$ are taken in the following forms,

$$a(t) = 2 - \frac{1+(1-\zeta t)\exp(\zeta(t-0.5T))}{(1+\exp(\zeta(t-0.5T)))^2}, \quad b(t) = \exp\left(\frac{5\pi^2 t}{1+\exp(\zeta(t-0.5T))}\right).$$

In the computation, the problem is considered on a mesh 41×41 and the final time $T = 0.02$. The dependence of $a(t)$ and $b(t)$ on ζ can be seen in Figure 1 and 2. It can be seen, for large values of the parameter ζ, $a(t)$ has a spike, while $b(t)$ approach discontinuous functions with a discontinuity point for $t = 0.5T$.

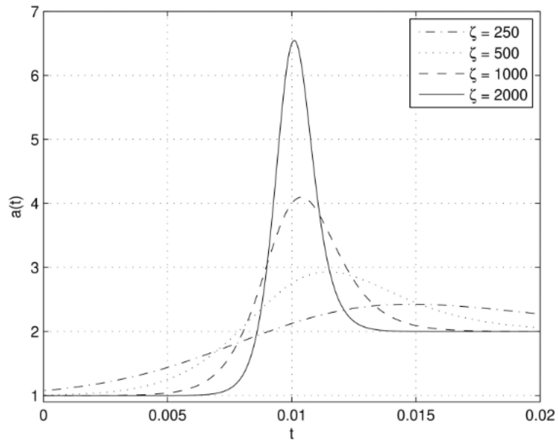

Figure 1. Solution of leading coefficient a(t).

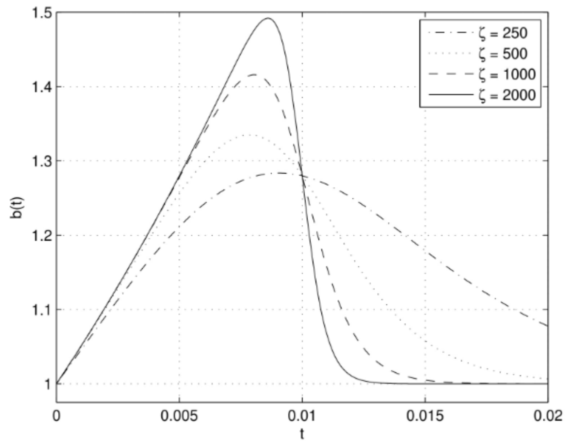

Figure 2. Solution of right hand side b(t).

It should be noted, to use the linearized scheme (5), we need to determine the values of $a(0)$. Actually, we can find $a(0)$ from the known conditions. Based on the known conditions in this example, for $t = 0$, we have,

$$u_t(\boldsymbol{x},0) - a(0)\Delta u(\boldsymbol{x},0) = b_0 f(\boldsymbol{x},0),$$

this equation satisfies all $\boldsymbol{x} \in \Omega$, and $\Delta u(\boldsymbol{x}, 0)$ can be obtained from initial conditions $u_0(\boldsymbol{x})$ by finite difference approximation, substituting $\boldsymbol{x}^*$(or $\boldsymbol{x}^{**}$) into the above formula, we obtain,

$$\Phi'(0) - a(0)\Delta u(\boldsymbol{x}^*,0) = b_0 f(\boldsymbol{x}^*,0),$$

then, we can get the approximation solution of $a(0)$.

According to the formula (9), it can be seen that the sign of Δu has uncertainty, which may lead to computational difficulties, so the selection of the observation points $\boldsymbol{x}^*$ and $\boldsymbol{x}^{**}$ in the calculation has an important influence on the result. In our example, notice that $\Delta u = 0$ when $x_1 = 0.5$ or $x_2 = 0.25, 0.75$, in order to get good results, we try to avoid taking these points as observation points. In the computation, we first solve the direct problem and use the obtained results as the values at the observation points for the solution of the inverse problem. The figure of the solution $u(\boldsymbol{x}, T)$ in direct problem is shown in Figure 3 with $N = 1200$ and $\zeta = 1000$.

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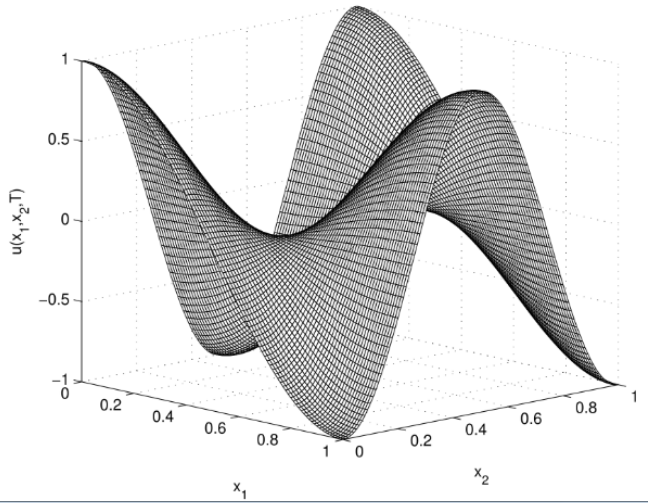

Figure 3. The figure of the solutions u(**x**, T) of the direct problem.

In the calculation of the inverse problem, we choose $\boldsymbol{x}^*$ = (0.4, 0.3) and $\boldsymbol{x}^{**}$ = (0.7, 0.8) as the observation points. The errors of leading coefficient $a(t)$ and right hand side $b(t)$ with different time steps are shown in Figure 4 and 5, respectively. We also consider the influence of the choice of the observation points, we select one point $\boldsymbol{x}^{**}$ = (0.9, 0.5), the other one $\boldsymbol{x}^* = (x^*_1, x^*_2)$. The influence of the choice of the observation point can be seen in Figure 6 and 7.

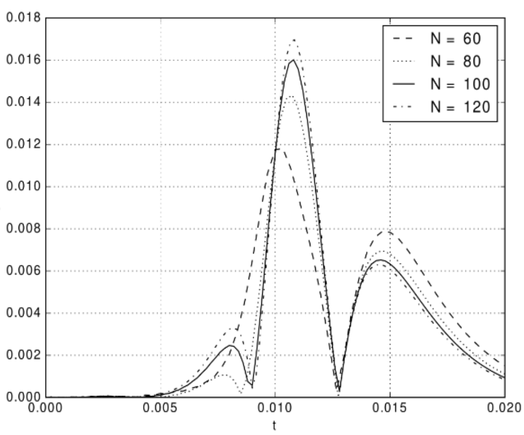

Figure 4. The errors $\epsilon_1(t)$ with different N.

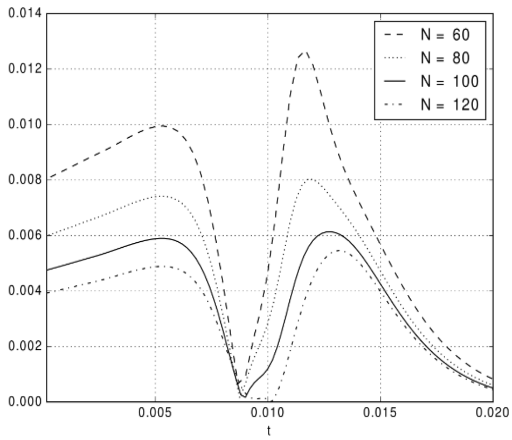

Figure 5. The errors $\epsilon_2(t)$ with different N.

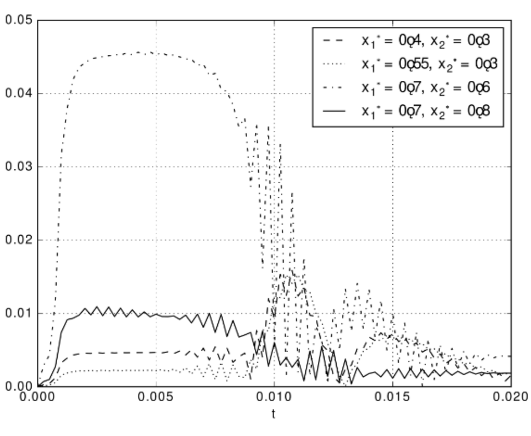

Figure 6. The errors $\epsilon_1(t)$ with different $\mathbf{x}^*$.

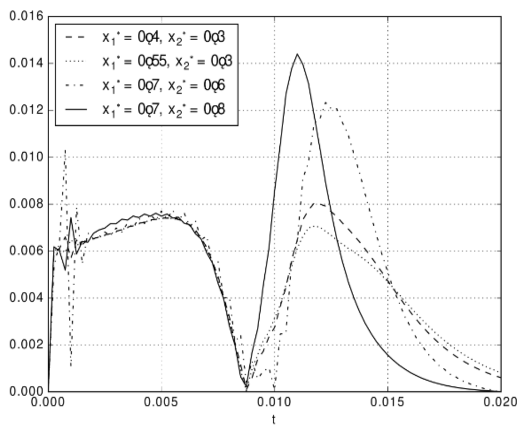

Figure 7. The errors $\epsilon_2(t)$ with different $\mathbf{x}^*$.

5. Conclusions

In this paper, a numerical method based on finite element method for solving the inverse problem of coefficient of simultaneously identify the leading coefficient and the right hand side function in a parabolic equation is proposed. To solve this nonlinear problem, linearized second-order schemes is used for linearizing the quadratic nonlinear term and combine with a decomposition of the approximate solution, at new time level the problem is transformed to three standard elliptic problems, numerical example demonstrates the efficiency of the numerical method.

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