IX All-Russian Student Olympiad in Mathematics with International Participation

1. Find

$$\lim_{t\to+\infty} t\sum_{k=1}^{+\infty} \frac{1}{k^2+t^2}.$$

2. Let $f(x) = 19^x - 13^x - 3^{2x} + 3^x$. Find All real roots of the equation $f(x) = 0$.

3. Prove that for any continuous non-decreasing function $g : [0,1] \to \mathbb{R}$ inequality

$$\frac{1}{2024}\int\limits_0^1 g(x)dx \leq \int\limits_0^1 x^{2023} g(x)dx,$$

is true. Find all functions for which equality will be true.

4. Find

$$\int\limits_L \frac{z^2}{z^3-1}\,dz,$$

where L — simple closed loop, such that $|Rez| + |Imz| = 1$.

5. A circle ω_0 of unit radius touches both branches of the parabola $y^2 = px$, $p > 0$. Based on ω_0, a sequence of circles ω_k is constructed such that ω_k externally touches the circle ω_{k-1} and both branches of the parabola, or the vertex of the parabola. Find the smallest value of n for which the circle ω_n touches the vertex of the parabola

6. All the numbers of $\prod\limits_{i=1}^{k} p_i^{m_i}$, are written in ascending order, where p_i is distinct fixed prime numbers, $m_i \in \mathbb{Z}_{\geq 0}$. Let $f_k(n)$ be a function that returns the n-th number on the board. Find the limit

$$\lim_{n\to\infty} f_k(n)^{n^{-1/k}}.$$

7. Let $\{a_i\}_{i=1}^k$ be natural numbers so that $1 \leq a_i \leq N$. In addition, it is known that $\sum\limits_{i=1}^{k}\sqrt{a_i}$ is not an integer. Let us denote that $\|x\|$ is the distance to the nearest integer. Prove that there is a constant c_k independent of N so that

$$\left\|\sum_{i=1}^{k}\sqrt{a_i}\right\| \geq c_k\ N^{1/2-2^{k-1}}.$$